Dartmouth College
Graduate Program in Mathematics

The Written Qual Book

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An Introduction

Welcome

Welcome to the "Official" Written Qual Book! If you're reading this, then you're probably a first-year graduate student at Dartmouth College and you're hoping for some idea of what's involved with the written qualifying exam. This book contains a wide variety of information, including historical information about the written exam, tips for getting through the first year of graduate school, and a collection of "complete" solutions to the previous written qualifying exams.

The Written Exam

The first step on your journey to candidacy and eventual graduation is passing the written qualifying exam. This section details some of the history of this test as well as the current format of the exam and ways to prepare for it.

1.2.1 The Exam Itself

The best source for official material about the qualifying exam is the graduate handbook which is available on the math department webpage. However, the information below should answer some of the "big questions":

- **What is the format?** The exam is broken into five topics exams, two separate exams in applied mathematics, and one in each of algebra, analysis, and topology. Each exam will consist of a 3-hour 6-question written exam and each student must pass three of the five topic exams by the end of summer to advance in the program.
- **When is it?** The exam *usually* takes place toward the end of June, typically the week before summer classes begin. The entire written exam has historically been split up over three days.
- **What topics are covered?** The specific material is determined by the content of the respective first year classes (101, 111, 103, 113, 106, 116, 126, 136, 104, and 114).

- **What resources are allowed during the exam?** No external resources are permitted on the exam. You must rely on the tools of the trade: pen(cil)s and paper.
- **When is the exam graded?** The problems are usually graded and scores are returned within a week or two of the exam date.
- **What happens if I fail?** It is worth noting that every year that the written exam has been offered, at least one student has failed the summer exam, so you are not alone. Currently, if you fail a section, you must retake and pass that section in the fall (just before the start of fall term) to remain in the program.

It is not necessary to completely answer every question to pass the qual and some credit may be awarded for partial solutions. However, this does not mean that you should simply write down everything that might be *somehow* relevant to the problem in an attempt to "score some points." The faculty members grading the exam are unlikely to be impressed by such an approach.

On the other hand, leaving a problem entirely blank does not send a particularly impressive message either. In the event that you are not able to make significant progress on a problem, it might be worthwhile to explain the approaches that you attempted and why they didn't work out. This shows your thought process without trying to slip something by the faculty.

1.2.2 History

Part of the "fun" of the qualifying exam is tracking the evolution of the format and question types over the years. Although the collective history of the written exam is not as entertaining as the details contained in The Qual Book[1], there are still some interesting facets to be examined.

The "olden" days

The current form of the qualification system was adopted beginning with the 2011-12 school year. The previous system did not have a written component. Instead, advancement to candidacy was contingent on successfully completing four oral exams. Specifically, each student had to pass oral exams in algebra, analysis, topology, and one additional topic of their choice. Students traditionally began taking oral exams during the spring term of their first year. The introductory courses for graduate students were also quite different as many of the course sequences were three quarters long instead of the current two. This explains the structure of some of the current first year courses (e.g., 103 mashing together measure theory and complex analysis into a single course and the lack of a general topology course).

Switching systems

Although the overall structure of the current qualification system (a written exam at the end of the first year and two oral exams during the second year) was put in place in 2011, the transition was not entirely smooth. The rules and procedures surrounding the written exam have been updated most years since it began. Some of the changes were relatively minor, whereas others were quite significant. We describe some of the steps in this evolutionary process below:

[1] If you're unfamiliar with this resource, one of the other graduate students is sure to have a copy somewhere. When oral qualifying exams come around, you might want to take a gander.

- **2012:** The first year of the exam is distinguished from later iterations in several notable ways. The most obvious difference is the fact that the questions were not separated by topic. Each day of the exam contained a mixture of question types. Additionally, the overall level of the questions on this exam is lower than many of the following years, particularly in analysis and topology. Many of the questions are derived from the basic knowledge sections of the previous system's practice qual questions.[2]
- **2013:** The big change for the second year was the separation of question types: each day of the exam featured material from only a single subject area.
- **2014:** It was clarified this year that students could be separated from the program after the summer exam.
- **2015:** Major procedural changes were introduced:
  1. All students in good standing[3] were guaranteed the ability to retake the exam in the fall if they failed in the summer.
  2. Students who failed portions of the exam were only required to retake the sections that they failed.[4]
  3. The policy that internal grades from the first year courses were to be given to the students at the conclusion of the course was reinforced.
  4. The graded exams were made available to students after the exam.
- **2016:** There were no significant updates to the qualifying exam rules this year. However, due to a situation where the instructor of a course was unable to write questions for their portion of the exam, another professor created the questions for the course.
- **2017:** The first applied math exam was given this year. Students were required to select 3 out of the 4 possible exams. Anyone who failed a section was asked to meet with faculty to discuss their performance.
- **2018:** This coming year, two distinct applied math exams will be given.

All of this goes to show that the written exam is *not* a polished product; it is still changing and updating. We expect this to continue.

1.2.3 Looking Forward

As the current composition of the department is evolving rapidly, further changes to the qualification system are being discussed by the faculty. We hope that this book will continue to be a valuable resource regardless of the changes made to the qualifying format.

[2] Look for these practice problems on the math department webpage.

[3] That is, you have not been told you're in bad standing.

[4] Prior to 2015, a student had to pass all three exams. If they were given a retake, regardless of how many they passed, they still had to pass all three exams.

1.2.4 Preparing for the Exam

There is no single "right approach" for studying the material in preparation for the written exam. Everyone learns and masters material differently.[5] That being said, some aspects of the preparation process are sufficiently generic and helpful to warrant inclusion in this document. Our main focus is providing suggestions about what material to study, how to study effectively, and what resources are most useful.

- The questions on the written exam are written by the faculty members that taught the respective first year course. Thus your class notes from those courses are (hopefully) your first and best source of study material. Additionally, homework and exam questions from those courses, as well as optional exercises presented in lecture[6], are also excellent places to start.

- Remember that the exam tests your ability to apply the knowledge that you gained in the courses to problems. Thus it is not sufficient to simply *know* the statements of results that were presented in class but, instead, understand how to apply them. Memorizing page after page of theorems, proofs, and examples should not be a substitute for understanding.

- As mentioned above, it would be foolish for us to try and offer general, prescriptive advice on how to study. Some people prefer to spend hours alone in silence revising their notes, while others prefer to discuss mathematical topics aloud and debate the best approaches to individual problems. However, there are a couple of things that are important regardless of your study method:

  (1) Preparing for this exam requires a significant amount of consistent, sustained effort. Do not put off your studying until the end of spring term. There are only a couple of weeks between when classes end and the qualifying exam. That is not enough time to review 9 months worth of material.

  (2) Make sure to take a step back and view the forest as well as the trees. There are many commonalities between the material presented in the first year courses and being able to make connections across different areas of mathematics is an important part of the preparations process.[7] Try to summarize the important ideas and concepts from each course and use that outline to guide your preparation.

- There are many different types of resources that can help you study for the exam. This book contains the problem statements and solutions to the previous written exams and can give you an idea of the general level and topics that have been included. As mentioned above, textbooks and written notes for the course are also a good place to start. Additionally, older graduate students can provide some insight into the problems that certain faculty prefer.[8]

- The single most useful thing you can do to prepare for the exam is to spend time with the faculty members teaching the first year courses; they have office hours for a reason.

[5] After about 16 years of formal schooling, we hope that you have a good sense for how you learn best.

[6] For instance, you might hear "... you can prove this on your own time." Maybe now is the time to do that.

[7] Sound familiar? Good! That means you read some of the graduate handbook.

[8] As with everything related to the written exam, the best source is the actual professor for the course in question. Specifically, some well-meaning eager-to-help graduate students may offer opinions that are not actually rooted in experience with the professor (or the course) in order to make you feel better. Exercise appropriate skepticism in all things.

All of the information in this section is superseded by the following directive: **talk to the professors**. Talk to them about the class material. Talk to them about the exam. Talk to them about how to prepare. Just talk to them. One of the best parts of the graduate program at Dartmouth is the accessibility of the faculty; make the most of it.

This Book

This book is intended as a record of the various written exams that have been given as part of the mathematics graduate program at Dartmouth College. While the problem statements are available through the math department's website, students over the years have been unsure of whether or not certain problems should be solvable for them. That is, given the flexibility that the faculty have with the course syllabi, some "optional" topics become essential to a particular iteration of the course and makes an appearance on the written exam.

By having a written record of solutions to the written exam problems, students will be able to look at the methods used to solve the problems and judge for themselves whether or not the material is related to what they have seen in class. Furthermore, by having complete solutions, this book can be a useful studying tool.

1.3.1 Current Edition

This is the second edition of the Written Qual Book. We have added the Summer 2017 exam and fixed solutions from the original version. No major stylistic changes have been implemented.

1.3.2 First Edition

This is the first edition of the Written Qual Book. We hope that the content addressed in the first three chapters will evolve as the preliminary exam does and that future years of graduate students will work as hard to make this resource useful and available to new students.

1.3.3 Format and Instructions

This book has been organized into three major sections: information, exam solutions, and exam commentary. The informational chapters are intended to answer questions about the written qualifying exam and offer generic advice. The exam solutions and commentary are split by subject, by year, and finally by problem. In all of this, there is one very important caveat: **expectations for the *same* problem will be different when given by a different instructor**. That is, each faculty member is responsible for determining how much detail is required. If your algebra professor doesn't care about the categorical definition of "natural," that's fine. However, if they do, you better include that in your solution.

Each problem has a complete statement of the question, as asked on the written qualifying exam, followed by a solution which was provided by one of the graduate students. Unsurprisingly, this solution should provide a detailed explanation of how the problem can be solved (occasionally with snarky footnotes). There is also a hyperlink to take you to the commentary associated to that problem, if any.

The commentary chapters serve two major purposes:

(1) It can be used to break a complicated problem into more conceptual pieces. For some problems, it is easy to get lost in the details; to lose sight of what is happening. In these cases (we hope), the commentary has extracted the essential proof and the details are found in the solution itself.

(2) It can provide context that would otherwise be missing. For instance, some non-intuitive problems were given as homework in the core courses and so it isn't as unreasonable for that group of graduate students to solve it on their written exam.

The commentary chapters have been purposefully separated from the solutions to keep the solution sections as "clean" as possible (for printing or specific types of studying).

1.3.4 **Disclaimers**

- This book was written by graduate students, for graduate students. As such, this book does *not* have any Official Stamp of Approval from the faculty.
- We recognize that typos and mistakes are a given, but we have done our best to proofread and evaluate the solutions provided in this book. When you find errors, please report them to the appropriate authorities.
- The vast majority of the solutions (from the first 10 exams) were written and subsequently edited by the authors. As such, please address comments, concerns, questions, and complaints to them.
- The solutions given will not always be the most elegant[9] but we hope that additional, perhaps more concise, solutions will be added to the commentary as time goes on.

1.3.5 **Acknowledgments**

We would generally like to thank the graduate students who have contributed solutions for this book:

- Class of 2017: Tim Dwyer, Everett Sullivan
- Class of 2018: Daryl DeFord, David Freund, Kate Moore, Justin Troyka
- Class of 2019: Ben Breen, Sara Chari
- Class of 2020: Juan Auli, Chris Coscia, Zach Garvey, Emma Hartman, Laura Petto, Lizzie Tripp
- Class of 2021: Victor Churchill, Sarah Manski, James Ronan

This book has been a wonderful opportunity for the graduate students to work together to create a resource for future classes.

[9]In fact, do not expect any of the solutions to qualify for being in "The Book."

2

Surviving Your First Year

Many people find the graduate program to be quite different than most of their previous schooling and the first year in particular can be quite an extreme transition. This chapter describes some of the integral components of the first year of the graduate program at Dartmouth and provides some commentary on graduate school in general.

First Year Outline

The graduate handbook[1] contains an outline of the first year in Section 2.1. In order to more strongly encourage you to read it, we do not reproduce that material here.

What is Graduate School?

A Ph.D. is a research degree and the purpose of graduate school is to train you to become an independent researcher and mathematician. By joining the graduate program, you are joining the community of mathematicians. There is a commensurate expectation that you will behave in a professional manner. Different aspects of this will be discussed in ethics and teaching assistant (TA) training but, in all of your actions, you should keep in mind that you are an adult mathematician representing Dartmouth College.

One of the tasks for your first year is deciding why you are here and what you want to do. The first year of this program is difficult and some aspects of this are even by design. Pursuing a Ph.D. is a complex task that trains you to be a research mathematician and requires a significant amount of independent work and effort.

Unlike an undergraduate program, *you* are responsible for your learning.[2] Thus, you must be willing to take ownership for your learning and progress through the program. One of the purposes of the first year is to force you to confront these issues. Mathematics research is a difficult career path and it requires a significant amount of dedication. Part of this year is learning if you are able/willing to commit to your own education and development.

[1] As you might have noticed, the authors of this document have a great deal of appreciation for the handbook. Read it.

[2] That is, you will probably have fewer people keeping tabs on you.

There is a common temptation among graduate students to stick with what is easy and familiar. For some, this means TA and teaching responsibilities; for others, it is coursework and problem sets. Do not let these aspects of the program keep you from your primary goal of writing a successful thesis.

More Details

In this section, we provide details about some common confusions that confront first year students. If you have any questions about topics not addressed here, people in the department are always happy to answer questions.

2.3.1 Classes and Grades

Perhaps one of the more obvious and natural components to the first year experience is that you will be taking classes.[3] These classes should provide some amount of familiarity (in terms of structure) to help you smoothly transition to graduate school. However, even in this, there are probably some unexpected differences and adjustments:

- Dartmouth's faculty have a propensity to incorporate Latin phrases in their lectures (e.g., *nota bene*, *a priori*, *a posteriori*). It's part of the faculty training program.
- Commutative diagrams (and category theory) are preferred to lines of equations.
- Problems can (and perhaps should) require you to apply material that is not directly taught in that course.[4] Since you are at Dartmouth, it is likely that you are used to getting excellent grades on *all* of your homework assignments in math classes. Thus the scores on your first sets of graduate assignments may come as a shock. This is a common situation and you can take comfort in the fact that you are surrounded by people who have had (and are having) similar experiences.

  If things are going well for you, that is fantastic but please remember that your classmates who might be struggling are your peers and you can help them without being condescending. Denouncing homework problems as "trivial" or "super easy" is not helpful commentary, regardless of the situation.

  Assignments in graduate school require a much greater level of independent reading and thinking than undergraduate work and making this transition is an important part of your first year experience. You will undoubtedly have to make changes to your studying methods and scheduling priorities (e.g., beginning an assignment the night before it is due is unlikely to remain an effective strategy). This is a *good* thing – graduate school is preparing you for something different than your undergraduate program and change is inevitable.
- Exams are frequently designed as learning experiences. For some courses, the exams will be week-long take-home assessments, full of complex multi-part problems.[5]

[3] This is not true of all graduate programs, certainly, but it's how undergraduate programs work everywhere.

[4] This is one reason that talking to faculty can be so helpful: they might point out a weak area that you never knew about!

[5] Please be considerate of your classmates during these stressful times. Walking around saying things like "question $\square$ is super easy" or "question $\triangle$ is impossible" is rude, unfair, and frankly unethical.

> Failing an exam in a core course is not necessarily like failing an exam in an undergraduate class. Your goal is to master the material in time to pass the summer exam and, for some topics, this simply takes longer than the four weeks before the first midterm or the ten weeks before the final. *The faculty members are aware of this.* While your professor (hopefully) designed the exam to be passable, your score on any one particular exam has little bearing on your final success in the program. If you "fail,"[6] you should certainly consult with the professor to see what you need to improve.

Even more significantly, grades are very different in Dartmouth's graduate program. As is standard, you will receive a letter grade for all of your hard work at the end of a course. For lecture courses, this will be one of the following: F (fail), LP (low pass), P (pass), or HP (high pass). A "P" indicates that you have done a sufficient amount of work to pass the class. This doesn't (necessarily) correlate with a "C" in a more traditional grading system, but it is the best analog.

If you receive an "LP" or an "F," you should consult faculty and the graduate handbook about what this means. These are unsatisfactory grades and can lead to immediate consequences for you. Do not take them lightly.

Pass is the normal grade. Even if you productively work on problems from dawn 'til dusk, turn in the most magnificent and elegant solutions, and ace the exams, some faculty simply do not assign high passes. Others do and they may use whatever criteria they feel is justified for doing so. Let us reinforce this (since it is something that takes a little bit of time to get used to): the letter "P" has *no* quantitative meaning. There is no particular average score cut-off that must be achieved to obtain it and trying to interpret these grades as anything other than decoration on a transcript is the road to madness. Similarly, the "HP" on your transcript gives you no further advantages than a pass would have.[7]

Since the grades reported to the college are not particularly useful in gauging your progression on the qual material, the department has instituted a set of internal grades for the 8 qual courses. These grades (on a scale of 1-5) are delivered to your mailbox sometime after the term ends and provide a few more details about how the course instructor feels about your readiness for the written exam. A low (or high) score here is not predictive but rather another piece of information to help you evaluate your progress.

2.3.2 Advising

As a new graduate student, you're inundated with different advisers. These individuals are supposed to assist you with the transition to graduate school *as you request.* Everyone has some silly questions about grad school when they start (and it's much better to get them answered than to let them remain mystifying). The two advisers you will interact with most commonly as a first year are described below:

- Adviser to graduate students: This individual has more of a "global" advising role. They are supposed to make sure you're doing all the different mandatory things that you're supposed to be doing (registering for classes, attending classes, settling into the department, applying for the NSF-GRP fellowship, etc.).
- Individual adviser: This individual may change (if you wish) and is "your" adviser for the year. They are supposed to help you navigate the courses for the year, help you make informed choices about what to do, and answer questions about life and times at Dartmouth.

[6] It is a little difficult to say exactly what this means given the relatively uninformative measure that is the LP/P/HP system (read on!).

[7] That's not to say that they're worthless. Certainly they provide an ego boost (deservedly or not).

2.3.3 Other Commitments

During the first year, there's more involved with graduate school than "take and pass classes." A lot more.

1. TA responsibilities: During your first year, you will TA (typically) during the fall and winter terms. Your standard responsibilities will include holding tutorial sessions from 7-9pm three nights a week and helping grade exams. It is tempting to devote significant amounts of time to your TA responsibility, especially since the material is familiar and the work feels doable. However, do not allow your TA preparation to compromise your own learning. Talk to the course instructors frequently – they like to know what is happening with their students.

   For some graduate students, this will not be a new experience (math center employment is fairly common for undergraduates). However, this is not the same. You are representing the department and, regardless of how it feels, there is a power differential that you should acknowledge.[8]

2. TA training: In the fall of your first and second years in the math program, you and your classmates will be meeting with a faculty member to understand your role as a teaching assistant. The syllabus for this training was developed in the math department and it helps to standardize the way that we support the students in our courses.

3. Ethics: As a first-year graduate student at Dartmouth, you are expected to take part in ethics training. This involves attending four 2-hour sessions throughout the term that cover different topics. Although the standard ethics curriculum is not really designed for mathematics students, there has been an attempt in recent years to update the syllabus to better reflect mathematical cultural norms.

4. GSS: The Graduate Student Seminar consists of talks by graduate students for graduate students (and free pizza). You should attend this seminar every week. It is a good chance to find out what the other graduate students in the department are doing and to get a feel for the research that happens here.

   In addition to the math and pizza, there is another important aspect to your involvement with GSS: supporting the other graduate students in the program. Graduate school can be difficult and encouraging your peers and classmates is very valuable. Historically, this department has a reputation for maintaining a supportive environment among the graduate students and the collaborative atmosphere fostered by the GSS program is part of that. Finally, it is likely that, at some point, you will also be in need of help or support yourself. Being an *active* part of the graduate community makes it much easier for other people to reach out to you.

5. Colloquium: The department colloquium brings in professors from other schools to give talks about their research. You are expected to attend this seminar every week. Even though the talks are usually advanced, it is an excellent opportunity to get exposure to many different research fields.

6. Gauss: You should setup an account on the department server early on. This will get you access to a math.dartmouth.edu e-mail account as well as a webpage.

7. CV: As a graduate student, you should maintain an updated CV.[9] Keeping track of your accomplishments as they happen is much easier than trying to remember them all when you are trying to put together a job application. Ask an older student for a template if you don't have one.

[8] Read the TA guidelines (available on the math department webpage!).

[9] Keeping an updated CV *and* webpage is even better.

8. NSF: During the fall term of your first year, you are eligible[10] to apply for the NSF Graduate Research Fellowship. The department has traditionally offered a series of workshops helping students to prepare their applications.[11]

9. Tea: The department holds tea every afternoon. Generally there is very little tea at this event, rather there are snacks and treats. This is a good chance to meet with students and faculty members and discuss a wide variety of topics (mathematical or otherwise). Similar to GSS participation, tea attendance is not only about food and math but also about being a part of the community of graduate students.

[10] Assuming U.S. citizenship.

[11] Mileage may vary.

3

Unsolicited Advice

This chapter consists of some advice collected from older graduate students concerning things that went well during their first year as well as things they wished had gone differently.

Classmates

In graduate school, you are part of a cohort (generally with 5 or so other people). Like any other work place, you don't choose the composition of this group. At Dartmouth, for your entire first year, you will be around them for classes, seminars, and tutorials. Hopefully you will enjoy their company and develop, at the very least, a quality working relationship with everyone. In reality, with all the time you're spending with these people, conflicts will naturally arise (and some of them may be your fault!).

Since we presume that everyone reading this is an adult, we will not go into managing interpersonal relationships. Instead, we will focus on how classmates can impact (positively or not) preparation for the written exam.

3.1.1 Collaboration

Your classmates are in a unique position that no one else in the department is in: they're also trying to prepare for the written exam. Even better, they're sitting in the same classes! This means that you and your classmates are in the best position to support each other through the introduction to graduate school and the first major hoop. It's likely that some (if not all) of you have been working on homework problems together anyway, so why stop there?

- **Scheduling:** However unfortunate it may be, people have different schedules, different study patterns, and different misunderstandings. Being responsible for 6 courses worth of material makes it naturally difficult to cover it all *as a group*. While collaboration may be the most productive method for you, it's also unreasonable to wait and try to do all of your studying in groups. *Too much collaboration can be detrimental.*

- **Group Size:** Working in small groups (of two or three), it's possible to address all the misunderstandings that come up with various concepts, examples, and proofs. Once you get to larger groups, it's much more likely that *someone is getting left behind*. This is a horrible feeling (whichever side of it you're on) but it's the only way that larger groups can maintain momentum.

- **Language:** When working with other people, there will (we hope) be times where you are helping as well as times where you are being helped. Especially when you're trying to assist someone else, it's *essential* to be aware of your use of language. For instance, in mathematics, we "all" know that the words "obviously," "trivial," and "clearly" should be used sparingly.[1]

  Somewhat less obvious (for some reason) is the fact that **you are a peer, not an instructor**. Don't talk to your classmates as though you're trying to teach them something. If a classmate has come to you for help, it's not a good time to be cryptic and you should *never* belittle their misunderstandings, misconceptions, or give them grief for not remembering a homework problem from 6 weeks prior[2].

- **Fears**: Working collaboratively can coalesce your concerns and fears into one giant doom bubble. You and your classmates are in the same boat: if you start to feel it sinking, it's natural that everyone else will pick up on those feelings too. To be a little more specific, if you find a subject that "everyone" is finding difficult, you might trap yourselves into aggrandizing it, possibly ignoring it, or otherwise spend many fruitless hours banging your head against it. The solution? Talk to *other* people.[3]

3.1.2 Competition

Dartmouth's math program is naturally non-competitive. We don't have kick-out quotas to meet, we're all in the same funding situation, and there are generally enough advisers to go around for our small program. And yet some amount of competition in qual courses is natural: you're trying to figure out "where you fit" as a mathematician and your peers become the most immediate and relevant comparison. As ever, this leads to some stupid ideas:

- "I'm spending so much more (less) time studying than $\square$."

  So what? Everyone has different methods of studying and different needs. Graduate school does require a significant amount of work but at the end of the day it is your accomplishments that will make the difference.

  At the same time, the amount of effort expended is usually a good first-order approximation to achievement. As with many of the other elements of this list, the key is not to focus on comparing yourself to others but instead to focus on your own progression.

- "My undergraduate background is not as good as everyone else's."

  Sometimes it can feel like everyone else has already seen large portions of the material in some of the first year courses and you are always struggling to catch up. Although this means that you may have to spend more time mastering the fundamentals of the material, it has little bearing on your ability to succeed in the program. You were admitted because the faculty believe that you can do it. Every year, students enter the program and pass the written exam without previous exposure to topics like point-set topology or module theory. Don't let this hold you back.

- "If I find a problem easy (hard), it must be easy (hard) for everyone!"

[1] If you hadn't heard this before, please take it to heart now.
[2] ...for example
[3] Hmm... wasn't there a professor for that course? I wonder what they have to say about this.

Okay, let's be fair here: no reasonable person thinks this way. However, this idea creeps into people's behavior in much more insidious ways. Just because you're finding a problem easy or difficult, you should be careful how you present the problem to other people.

Imagine, if you will, that someone breezed though a problem in 5 minutes but you had spent *hours* on it. How would you feel about them telling you that it was "so easy" and you should have: (a) asked for help, (b) done the problem more intelligently, or (c) not wasted your time?[4] In all such things, please **be conscious of how you treat other people**.

While we're on the subject, we should note that this not only applies to qual questions but also homework and exams. With take-home exams in particular, some people have taken to wandering around the department, mentioning the relative difficulties of problems *while the exam is still being taken*. This is not only unethical (in fact, a violation of most take-home exam policies) and unfair, it's also extremely rude. Please put some care into how you talk about the difficulty of problems in *all* situations.

- "I'm getting higher marks than □, so I'm doing well in the program."

  Did you read the section on grades? Go back and think about what you've done. Grades (for a course, test, or assignment) tell you very little about anything. They can underline difficulties and misunderstandings that you're having, but they don't tell you how well you actually understand the material.

- "I'm getting some of the lowest marks, so I'm failing the program."

  Nooooooo, stop! Reread the section on grades, please! Whatever grades you're getting, you can dedicate serious effort to preparing for the written exam (talk with peers and faculty, read books). However you feel about your grades, they do not dictate your performance on the written exam. For most people, math takes time to learn and you may be surprised how much more makes sense *after* a core course is over.

- "Something great happened for one of my classmates, now I'm jealous/worried that I'm falling behind."

  During your time at Dartmouth, your classmates will (hopefully) succeed at things, publish papers, win grants, and get hired for exciting positions. There probably isn't anything that we can write here to help with jealousy. However, it is worth reinforcing that everyone moves at their own pace and trying to compare yourself to each accomplishment of each of your peers is not likely to be a productive activity. Celebrate their achievements and remember to act gracefully when you succeed.

Common Frustrations

3.2.1 The "Quality" Debate

One of the more obvious frustrations that occurs in the mathematical community (and Dartmouth's math graduate program is no exception!) is the discussion of which field of mathematics is "best." So we pit

[4] These are all based on actual events experienced by *multiple* graduate students. It happens and it's not okay.

pure math versus applied math, combinatorics versus "real" math, and so on. These comparisons are all absurd, pointless, and divisive.

Many graduate students in $\square$ have put themselves in some sort of ridiculous group *opposed* to $\triangle$, taking pride in an ignorance of $\triangle$ (suggesting that $\triangle$ is easy or worthless). Honestly, you can fill $\square$ and $\triangle$ with any pair of fields (e.g., pure and applied math). This is also absurd, pointless, and divisive; ignorance and attempting one-upmanship are never positive qualities.

3.2.2 Older Graduate Students

It is generally understood that, despite their best intentions, older graduate students occasionally offer remarkably unhelpful advice. This can range from comments that have no basis in reality ("everyone passes"[5]) to unhelpful platitudes ("it will be ok") to uninformed assessments of your progress ("you are doing well"). Obviously not everyone passes (**at least one student has failed the summer exam every year**), it isn't always ok, and other students rarely have any real indicators of how you are doing. The first year is exceptionally stressful[6] and putting up with this kind of nonsense doesn't help.

The older graduate students earnestly want you to succeed. Most of them even want to help support you through the first year experience. However, intent is not always relevant and, when you're at wit's end, the last thing you need is someone telling you that your concerns or fears are unjustified or unimportant. The best advice we can offer is to ignore the perpetrators.[7]

Some especially jaded graduate students will tell you how unimportant the qual courses are for *real* research and how they haven't touched the material from the core courses in years. We believe this is meant to be encouraging. After all, regardless of their importance, you will have spent a year working intensely and the payoff will be that you can immediately forget all of that information! Isn't that wonderful? So let's talk about this for a moment:

- The qual courses have been chosen to, in the view of the faculty, address material that they believe all Ph.D. mathematicians should know.[8] This should be immediately and viscerally dissociated from the notion of being essential for *all* mathematical research. These things are incomparable.

- If you aren't using the material from the core courses then, yes, it's going to fade. However, despite how some people talk about it, this is not a matter of pride. The more fundamentals and tools you have at your disposal, the better you will be able to connect disparate ideas.

- Being able to speak intelligibly with other mathematicians with different interests (including your classmates) is a feature, not a bug.

- Finally, you never know what classes you will be asked to teach down the road or how your research areas will change.

[5]The authors will never forget being told by an older student (who had failed the exam) that nobody had ever failed the exam. Not helpful.

[6]For most people. There are a decided few who make it through unscathed. If this applies to you, understand that you are in the minority and consequently need to be even *more* compassionate toward those who are suffering.

[7]Actually, it would be nice if you told them how their commentary isn't wanted or helpful. At the same time, we realize that this sort of confrontation is the last resort for some people.

[8]Individual faculty opinions may vary.

Common Crises

There are many crises that naturally arise during the first year of graduate school.[9] This is normal, possibly even an important part of the process. After all, grad school is a huge life change. On top of the stress of all that entails, you're also trying to establish yourself as a professional mathematician (and determining whether that's what you want!). Below is a list of common crises:[10]

- **Can I "make it?" Am I smart enough for this?**

  In the eyes of the faculty, yes. You were accepted to the program because the faculty believe that you will be able to succeed in Dartmouth's program. You may doubt their confidence, but know that this is a common occurrence.[11] While having "smarts" will get you through a lot of graduate school, dedication is much more valuable and will carry you through equally well.

  If you really want to make it through the program, you can find a way to make it happen. Everyone has their unique obstacles to work through and so there's no easy solution to this concern. Nonetheless, there are people (fellow graduate students, faculty, etc.) who will help – you don't have to do this alone.

- **Is graduate school the correct choice for me?**

  Graduate school isn't for everyone. Between the extra years of schooling, generally smaller paycheck, and workload, lots of people would rather choose a different career path. Starting graduate school and coming to this conclusion is not the end of the world. Frankly, most of the graduate students have wondered whether or not they should quit at one time or another. Be honest with yourself and decide what you want.

  It is not uncommon for graduate students to find new interests during graduate school (e.g., discovering a passion for teaching or research) or to grow weary of certain aspects of academia (e.g., teaching or research). There is nothing wrong with this and it is always better to learn these things earlier than later. There isn't any way to know if you will want to enjoy the unique balance of responsibilities and tasks that accompany graduate study until you try them.

  If you have a specific career in mind (whether that involves teaching, academia, industry, or something else entirely), you should think about whether graduate school is actually required or if it's something you simply want for yourself. Actually, instead of just thinking about it, talk to people in that profession or find more information online. Regardless, you should consult with many faculty members at every stage of your development. They are here to help and they're generally happy doing so!

- **I don't like math as much as everyone else.**

  That's okay. No, really, it's okay. Presumably you're in graduate school for math because you enjoy math, but nothing says that it has to be your entire life. Even though it's probably easier to get through the trials of graduate school if doing math is something that you actively want to do, you can make it through by consistently dedicating time to whatever needs doing.

[9] These are not actually exclusive to the first year, but that's our focus for this book.
[10] In the spirit of this chapter, they are accompanied by commentary!
[11] The authors certainly had this experience and we know that many others did/do as well.

It can be difficult to discuss this topic with most of the graduate students; as a whole, they seem to have a passion for math and want to do it all the time (some even say that it's what they do for downtime!). Nonetheless, you are not alone and you should try to find someone with whom you can discuss this concern.

Make sure you take time for things that you enjoy (reading, sports, video games, knitting[12], etc.), especially if they're important to you. Dropping the important aspects of your life to pursue math is going to leave you feeling unsatisfied. However, if you want to make it through graduate school, you're going to have to put consistent time and energy into math (just don't let it become "everything").

- **What am I doing here? Do I deserve to be here?**

  You're here and, while it's possibly unsatisfactory, that's as close as you're going to get to "deserving" to be here. Without delving into a philosophical debate about some of the formal definitions[13] it is tough to offer a substantive answer to this question. That being said, this is a common concern among graduate students and most of the other students in the program have confronted similar worries about themselves at various times in the program.

- **Why does everyone else have their life together?**

  They don't. People project the image that they have life figured out because they don't want to be seen as someone who *doesn't* have their life figured out. There are always hurdles to overcome, dramas to handle, and problems that couldn't have been predicted. Take it one day at a time and try to juggle as many balls as you can.

  All this being said, it's hard to see "everyone else" succeeding if you don't feel as though you are. This is especially true with research where progress might take months or longer. The trick is to push forward despite not having everything figured out.

- **How do I manage all of this work?**

  With graduate school, there's always a lot to do. Whether that's because of classes, TA responsibilities, seminars, or *simply doing something for yourself*, time gets used quickly. It's important to prioritize and keep track of what needs to be done. It's all too easy to fall into the trap of getting overwhelmed and not doing *anything*. However, you're not alone in this – talk to other graduate students, talk to your adviser, talk to your professors. You never know who might have an insight that will help you make it all work.

- **I know what I need to do to prepare for and pass the qual, but there's not enough time!"**

  Even in the best of circumstances (i.e., you know what you need to do, how to do it, and *you are actually doing it*), it's normal to feel that your effort is never enough; that it's simply impossible be *actually* prepared for the exam. This is entirely correct.

  There's always more that we can/should/must study in preparation for the qual. It doesn't matter if you're following all of our advice from the section on preparing for the exam, you will probably still feel that you're behind (perhaps horribly so). The trick is to realize that most people walk into

[12] Or crocheting.

[13] "Deserve," for example.

the exams feeling under prepared – you're definitely not alone. Try not to let this despair control you; keep plugging away at the material and simply try to be as prepared as you can.[14]

- **What happens if I fail?**[15]

  Failing happens. Since the inception of the written exam, at least one person has failed it every year. This doesn't mean that they weren't fit for graduate school, it just means that they needed to work on the "core" material for longer to have it gel. While no one likes to fail, everyone (under the current version of the exam) gets a second chance – show the faculty that you can make substantial progress and trounce the exam before the end of summer term.

  If you've failed the exam, you're going to want to meet with faculty frequently over the summer (for whatever sections are relevant). They will meet with you and help, but you need to hold up your end as well. This is a great opportunity to demonstrate that you can master material on your own. There is a tendency for students to want to decelerate after a long year, but being fully prepared for the fall exam is a very important benchmark. As addressed more directly in the next question, the summer after your first year is not a break from the program but rather a chance for you to work to develop your mathematical skills and interests.

  Although the current system does guarantee that you will be allowed to retake sections of the qualifying exam that you do not pass the first time, do not simply ignore one topic in hopes of passing the other two and mastering the third over the summer. This type of "gaming" the system does not reflect well on you as a student and runs counter to the ethos of the graduate program.

- **What happens if I pass?**

  Most practically, the answer to this question is that it is time to begin to study for your oral quals. There are few delineated responsibilities during the summer after your first year. You *should* take some time to relax after the exam. However, one of the most common mistakes that students make is to confuse the lack of immediate tasks with "nothing-to-do." This is a perfect time to begin speaking with possible advisors, forming your qual committees, and preparing for the oral exams. Is it possible to be successful without doing too much work over the summer term? Probably, but why are you wasting what could be productive and beneficial time?

- **What is the point of this exam? It clearly isn't a perfect measure of my abilities.**

  You're right, it isn't. No method of assessing students is perfect and the written qualifying exam is not (and will never be) an exception to this rule, regardless of the best intentions of the faculty. It is simply too difficult to measure knowledge, effort, and potential ability. Especially during the first year at Dartmouth, we should acknowledge this directly and, while you should work hard to learn the material (or pass the exam), you should *not* let the results of the exam define you. We should also not allow ourselves to judge others by their results in these exams; passing and failing can mean so many different things for each person that this is pointless.[16]

[14] Spending every waking minute on it is a bad idea too. Take some time for yourself.

[15] The authors have agreed to open a gelateria in Italy if they are separated from the program.

[16] And not very compassionate.

Written Qualifying Exam: Problems and Solutions

4

Algebra

Algebra Exam

Summer 2012

Problem 4.1.1

Ideals and quotients.

(a) Find all ideals of the quotient ring $\mathbb{Q}[x]/\langle x^{14}-1\rangle$. In particular, how many such ideals are there?

(b) Determine the structure of the quotient ring $\mathbb{Z}[x]/\langle 5, x^2-2\rangle$. Be as precise as you can.

Notes and Comments

Proof of (a). First, we note that

$$x^{14}-1 = (x-1)(x+1)(x^6+x^5+\cdots+1)(x^6-x^5+x^4-\cdots+1) = \Phi_1\Phi_2\Phi_7\Phi_{14}.$$

By the Correspondence Theorem, any ideal of the quotient is an ideal in the base ring containing the quotient ideal. Since $\mathbb{Q}$ is a field, $\mathbb{Q}[x]$ is a PID and hence, up to units, any ideal can be defined by a single generating polynomial. Thus, we can represent the ideals containing $f = x^{14}-1$ as one of the $2^4 = 16$ products of the irreducible cyclotomic polynomials Φ_1, Φ_2, Φ_7, and Φ_{14}, including the empty product (all of $\mathbb{Q}$). □

Proof of (b). Observe that $\mathbb{Z}[x]/\langle 5, x^2-2\rangle \cong (\mathbb{Z}/5\mathbb{Z})[x]/\langle x^2-2\rangle$. We can check by direct computation that x^2-2 has no roots modulo 5

$$0^2-2 \equiv -2,\ 1^2-2 \equiv -1,\ 2^2-2 \equiv 2,\ 3^2-2 \equiv 2,\ 4^2-2 \equiv 4.$$

and hence x^2-2 is irreducible over $\mathbb{Z}/5\mathbb{Z}$. This implies that the quotient is isomorphic to $\mathbb{F}_{25}$ since all non-trivial quadratic extensions of prime order fields are isomorphic. □

Problem 4.1.2

Let L be the splitting field over $\mathbb{Q}$ of $x^9 - 8$.

(a) Determine the degree of $[L : \mathbb{Q}]$ carefully, explaining all conclusions.

(b) Justify whether or not the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is abelian.

(c) Justify whether or not the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is solvable.

Notes and Comments

Proof of (a). Let ζ_9 denote a primitive 9th root of unity. Then the roots of $x^9-8 = (x^3-2)(x^6+2x^3+4)$ are $\zeta_9^k \sqrt[9]{8}$ and hence $L = \mathbb{Q}(\zeta_9, \sqrt[3]{2})$. Note that L is the compositum of $\mathbb{Q}(\zeta_9)$ and $\mathbb{Q}(\sqrt[3]{2})$ which, respectively, have degrees $\varphi(9) = 6$ and 3 since $\mathbb{Q}(\zeta_9)$ is cyclotomic and $\varphi(9) = 6$ and (x^3-2) is irreducible by Eisenstein. We observe that $\mathbb{Q}(\sqrt[3]{2})$ is not a normal extension of $\mathbb{Q}$ since it is a real extension but the other two roots of $(x^3 - 2)$ are complex.

By multiplicativity of degrees, $\mathbb{Q}(\zeta_9) \cap \mathbb{Q}(\sqrt[3]{2})$ is either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt[3]{2})$. Since $\mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{R}$ is not a normal extension, it is not a subextension of $\mathbb{Q}(\zeta_9)$ (which is abelian) and hence $\mathbb{Q}(\zeta_9) \cap \mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}$. Thus the entire extension has degree 18 by multiplicativity in towers.

$$\begin{tikzcd}
 & L = \mathbb{Q}(\zeta_9, \sqrt[3]{2}) \arrow[dl, dash, "3"'] \arrow[dr, dash, "6"] \arrow[dd, dotted, dash, "18"'] & \\
\mathbb{Q}(\zeta_9) \arrow[dr, dash, "6"'] & & \mathbb{Q}(\sqrt[3]{2}) \arrow[dl, dash, "3"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$\square$

Proof of (b). The Galois group is not abelian since $\mathbb{Q}(\sqrt[3]{2})$ is not normal. $\square$

Proof of (c). The extension is solvable since we can first adjoin the cyclotomic extension and then the $\sqrt[3]{2}$. $\square$

Problem 4.1.3

Let the field K be an extension field of a field k. Show that there is a natural isomorphism of K-algebras $K \otimes_k M_n(k) \to M_n(K)$, where for a ring R, $M_n(R)$ denotes the ring of $n \times n$ matrices over R.

Notes and Comments

Proof. Define the map $\varphi : K \otimes_k M_n(k) \to M_n(K)$ on elementary tensors as $\varphi(a \otimes M) = aM$.[1] This takes a basis $\{1 \otimes E_{i,j}\}$ to a basis $\{E_{i,j}\}$ and hence is an algebra isomorphism.

[1] More precisely, define a map on $K \times M_n(k)$ in this way. Then it is k-balanced and k-linear and thus uniquely extends to a map defined on elementary tensors in this way.

To show that this is natural isomorphism, we must show that a specific diagram commutes. Let $F_1, F_2 : \{k\text{-algebras}\} \to \{K\text{-algebras}\}$ be [2]functors defined by $F_1(Z) = Z \otimes_k M_n(k)$ and $F_2(Z) = M_n(Z)$. Let L and J be k-algebras and $f : L \to J$ a k-algebra homomorphism. For any $\ell \in L$ and $M \in M_n(k)$, we have $F_1(f)(\ell \otimes M) = f(\ell) \otimes M$ and $F_2(\ell M) = f(\ell)M$ by k-linearity. Since

$$\varphi_J(F_1(f)(\ell \otimes M)) = \varphi_J(f(\ell) \otimes M) = f(\ell) \cdot M = F_2(f)(\ell \cdot M) = F_2(f)(\varphi_L(\ell \otimes M)),$$

the following diagram commutes

$$\begin{tikzcd}
\ell \otimes M \arrow[rrr, mapsto, bend left] \arrow[ddd, mapsto, bend right] & & & \ell \cdot M \arrow[ddd, mapsto, bend left] \\
& L \otimes_k M_n(k) \arrow[r, "\varphi_L"] \arrow[d, "F_1(f)"'] & M_n(L) \arrow[d, "F_2(f)"] & \\
& J \otimes_k M_n(k) \arrow[r, "\varphi_J"'] & M_n(J) & \\
f(\ell) \otimes m \arrow[rrr, mapsto, bend right] & & & f(\ell) \cdot M
\end{tikzcd}$$

and thus proves the naturality of φ as desired. $\square$

Problem 4.1.4

Let T be a linear operator on a finite-dimensional vector space V defined over a field k. Let $\chi_T(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_r)^{m_r}$ be the characteristic polynomial and assume all the λ_i are distinct. Let V_i be the eigenspace corresponding to the eigenvalue λ_i.

(a) Show that $\dim V_i \geq 1$ for all i, $1 \leq i \leq r$.

(b) Choose nonzero $v_i \in V_i$. Show that $\{v_1, \ldots, v_r\}$ is linearly independent.

(c) Conclude that if $\dim V_i = m_i$ for all i, then T is diagonalizable.

Notes and Comments

[2] (capitalization-)

Proof of (a). We assume throughout that $m_i \geq 1$. View V as a $k[x]$-module and call it V_T. Then, V_T is a finitely generated torsion module over a PID and so, by the Primary Decomposition Theorem, V_T uniquely (up to reordering) decomposes as

$$V_T = M_1 \oplus \cdots \oplus M_r$$

where M_i is the $(x - \lambda_i)$-primary submodule of M. That is, M_i is the direct sum of generalized λ_i-eigenspaces. (So each summand is cyclic and hence contains an eigenvector of T with eigenvalue λ_i.)

As $m_i \geq 1$, $\dim M_i \geq 1$. Hence V contains an eigenvector of T with eigenvalue λ_i and $\dim V_i \geq 1$. □

Proof of (b). This is immediate since the different $(x-\lambda_i)$-primary submodules have trivial intersection. □

Proof of (c). If $\dim V_i = m_i$, then M_i splits as a direct sum of m_i 1-dimensional submodules, each corresponding to an eigenvector. Hence V_T is the direct sum of 1-dimensional spaces and hence the Jordan Canonical Form J of T consists of a diagonal matrix. As $[T]$ is similar to J, T is necessarily diagonalizable. □

Problem 4.1.5

Show that any group of order 30 is the semidirect product of two smaller *abelian* groups.

Notes and Comments

Proof. Denote the s-Sylow subgroups by P for $s = 2$, Q for $s = 3$, and R for $s = 5$. Notice that, by counting subgroups with the Sylow theorems, there must be either a unique (normal) subgroup of order 5 or of order 3.[3] Thus there is always a subgroup ($QR = RQ$) of order 15 in G which has index 2 and is hence normal.[4] There is a unique group of order 15 ($\mathbb{Z}/15\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$) and a unique group of order 2 ($\mathbb{Z}/2\mathbb{Z}$). Thus, by the internal semi-direct product criterion, any group of order 30 can be realized as $\mathbb{Z}/15\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ which has abelian components. □

Problem 4.1.6

Let $m > 1$ be a square-free integer and $n \geq 1$ an *odd* integer. Let $F/\mathbb{Q}$ be any field extension with $[F : \mathbb{Q}] = 2$. Show that $x^n - m$ is irreducible in the polynomial ring $F[x]$.

Notes and Comments

Proof. Since $m > 1$ is square-free, there exists a prime p such that p divides m and p^2 does not divide m. Therefore, by Eisenstein's Criterion, $x^n - m$ is irreducible over $\mathbb{Q}$. It follows that $[\mathbb{Q}(\sqrt[n]{m}) : \mathbb{Q}] = n$. Taking the composite, we have $\mathbb{Q}(\sqrt[n]{m})F = F(\sqrt[n]{m})$ and $[F(\sqrt[n]{m}) : \mathbb{Q}]$ divides $2n$. Since n and 2 are relatively prime (because n was assumed to be odd), and both n and 2 must divide $[F(\sqrt[n]{m}) : \mathbb{Q}]$, we have $[F(\sqrt[n]{m}) : \mathbb{Q}] = 2n$. Therefore, $[F(\sqrt[n]{m}) : F] = n$. Since $x^n - m$ is a monic polynomial of degree n with root $\sqrt[n]{m}$, $x^n - m$ is the minimal polynomial for this extension. Hence, $x^n - m$ is irreducible in $F[x]$. □

[3] A more general statement is proved in complete detail in Fall 2012 Algebra Exam problem #4.

[4] As 2 is the smallest prime dividing $|G| = 30$.

Algebra Exam

Fall 2012

Problem 4.2.1

Let L be the splitting field of $x^{15} - 8$ over $\mathbb{Q}$ and let G be the Galois group of $L/\mathbb{Q}$. Show that G is a semi–direct prodcut of two proper subgroups K and H. Identify K and H by their intermediate fields and determine their isomorhpism types.

Notes and Comments

Proof. The roots of the polynomial take the form $\zeta_{15}^k \sqrt[5]{2}$ so $L = \mathbb{Q}(\zeta_{15}, \sqrt[5]{2})$. We begin by drawing the standard Galois diagram and notice that the total degree follows from multiplicativity in towers since $\varphi(15) = 8$ and 5 are relatively prime:

$$\begin{tikzcd}
& L \arrow[dl, dash, "5"'] \arrow[dr, dash, "8"] & \\
L^H = \mathbb{Q}(\zeta_{15}) \arrow[dr, dash, "8"'] & & \mathbb{Q}(\sqrt[5]{2}) = L^K \arrow[dl, dash, "5"] \\
& \mathbb{Q} &
\end{tikzcd}$$

Let H and K be the respective subgroups of G associated with the fixed fields L^H and L^K labeled in the figure. As $L^H/\mathbb{Q}$ is a cyclotomic extension, H is normal in G. We must have $L^H \cap L^K = \mathbb{Q}$ since $\gcd(5, 8) = 1$ and $L^H L^K = L$ by the definition of the compositum. So, by the Galois correspondence, $HK = G$ and $H \cap K = \{1\}$. Thus $G \cong H \rtimes K$ by the semi-direct criterion.

We selected H and K with the Fundamental Theorem of Galois Theory to fix the intermediate fields and there is only choice for K since there is only one group of order 5. From cyclotomic theory, we know that $H \cong (\mathbb{Z}/15\mathbb{Z})^\times \cong \mathbb{Z}/8\mathbb{Z}$ since $3 \not| (5 - 1)$. $\square$

Problem 4.2.2

Give three equivalent conditions which characterize when an algebraic extension of fields L/K is a normal extension and prove any two are equivalent.

Notes and Comments

Proof. Let $\overline{K}$ be the algebraic closure of K. The following are equivalent:

(1) Every embedding $L/K \to \overline{K}$ is an automorphism of L.

(2) L is the splitting field of a family of polynomials in $K[x]$.

(3) Every irreducible polynomial in $K[x]$ with one root in L splits in L.

$\underline{(1) \Rightarrow (2, 3)}$: Let $\alpha \in L$ and $p = m_{\alpha,K}$. For any root β of p in $\overline{K}$, there exists an embedding $\tau : K(\alpha)/K \to K(\beta) \subseteq \overline{K}$ which maps $\alpha \mapsto \beta$. By Theorem V.2.8 (Lang), this extends to an embedding

$\sigma : L/K \to \overline{K}$ with $\sigma\alpha = \beta$. By (1), σ is an automorphism of L and thus every root of p lies in L. Thus we have (3).

Moreover, L is the splitting field of the polynomials $\{m_{\alpha,K} \mid \alpha \in L\}$ and hence (2). This is contained in L by what we did above.

(2) $\Rightarrow$ (1): Let $\sigma : L/K \to \overline{K}$ be an embedding. Let L be the splitting field of a family $\{f_i\}_{i\in I}$ of polynomials in $K[x]$. If $\alpha \in L$ is a root of some f_i, then $\sigma\alpha$ is another root of f_i and hence it's in L.

Since L is the splitting field of this family of polynomials, L is generated over K by the roots of the f_i's. As σ preserves the algebraic operations, we have $\sigma(L) \subseteq L$. By the Fundamental Lemma[5] (Lang), $\sigma(L) = L$ and hence (1).

(3) $\Rightarrow$ (1): Let $\sigma : L/K \to \overline{K}$ be an embedding. Let $\alpha \in L$ and $p = m_{\alpha,K}$. Thus $\beta = \sigma\alpha$ is a root of p in $\overline{K}$. By assumption (3), we have $\beta = \sigma\alpha \in L$ and hence $\sigma(L) \subseteq L$. So by the Fundamental Lemma, we have $\sigma(L) = L$. □

Problem 4.2.3

Let F be a field of characteristic 0, $f \in F[x]$ an irreducible polynomial of degree $n \geq 1$ and K the splitting field of f over F. It should be well-known that $[K : F] \leq n!$. The point of this problem is to show $[K : F] \mid n!$. Hint: Prove that there exists an injective homomorphism $\mathrm{Gal}(K/F) \to S_n$ where S_n is the symmetric group on n letters.

Notes and Comments

Proof. Since F has characteristic 0, K is a finite separable extension of F. Hence the separable degree of K/F is n and $S = \{\xi_1, \xi_2, \ldots, \xi_n\}$ are the distinct roots of f in K. Since the Galois group $G = \mathrm{Gal}(K/F)$ takes roots of f to roots of f, we have an action of G on the set S; hence a homomorphism $\varphi : G \to S_n$ given by $\varphi(\sigma)$ being the permutation defined by $\xi_i \mapsto \sigma(\xi_i)$.

Consider the kernel of this permutation homomorphism. If $\varphi(\sigma)$ is the identity, then σ fixes all roots of f. Hence σ fixes all of $K = F(\xi_1, \xi_2, \ldots, \xi_n)$. Thus σ is trivial and so φ is injective. Thus $G \cong \varphi(G)$ by the First Isomorphism Theorem. That is $|G| = |\varphi(G)|$ must divide $|S_n| = n!$ by Lagrange's Theorem. □

Problem 4.2.4

Let G be a group of order pqr, where $p < q < r$ are distinct primes. Show that G is solvable.

Notes and Comments

Proof. Let n_p, n_q, and n_r denote the number of p, q, and r-Sylow subgroups, respectively, and let P, Q, and R be arbitrarily chosen p, q and r-Sylow subgroups. Observe that $|P| = p, |Q| = q$ and $|R| = r$ since $|G| = pqr$. We will start with a standard counting argument to show that at least one of the Sylow subgroups must be normal.

The Sylow theorems tell us that $n_r \equiv 1 \pmod{r}$, and $n_r | [G : R] = pq$. So $n_r \in \{1, p, q, pq\}$. Since $p < q < r$, we can't have $p \equiv 1 \pmod{r}$ or $q \equiv 1 \pmod{r}$, so $n_r = 1$ or pq. If $n_r = 1$, then the r-Sylow subgroup is normal and we're done. Otherwise, $n_r = pq$ and we have $pq(r-1)$elements of order r.

Similarly, $n_q \equiv 1 \pmod{q}$ and $n_q | [G : Q] = pr$. So $n_q \in \{1, p, r, pr\}$. Again, we can't have $p \equiv 1 \pmod{q}$ since $p < q$ and so $n_q \in \{1, r, pr\}$. If $n_q = 1$, the q-Sylow subgroup is normal and we're done. Otherwise, n_q is p or pr, giving us either $p(q-1)$ or $pr(q-1)$ elements of order q.

[5] Any embedding $\sigma : L \to L$ over K is an automorphism of L.

Also by similar arguments, n_p is $1, q, r$ or qr. If P is not normal, we have either $q(p-1), r(p-1)$ or $qr(p-1)$ elements of order p.

If none of the Sylow subgroups are normal, then we have at least $pq(r-1)$ elements of order r, at least $p(q-1)$ elements of order q and at least $q(p-1)$ elements of order p. That is, we have at least

$$pqr - pq + pq - p + pq - q + 1 = pqr + (p-1)(q-1) > pqr$$

elements in G (including the identity). $\lightning$ Thus, one of the Sylow subgroups is normal.

Claim: At least one of Q or R must be normal in G.

Proof. If $P \trianglelefteq G$ and Q and R are not normal, then $n_p = 1$, $n_q = r$ or pr, and $n_r = pq$. In this case, we have $p-1$ elements of order p, at least $r(q-1)$ elements of order q and $pq(r-1)$ elements of order r. Together with the identity element, this gives at least

$$p - 1 + qr - r + pqr - pq = pqr + q(r-p) + p - 1 > pqr$$

elements in G since $r > p$. $\lightning$ Therefore either $Q \trianglelefteq G$ or $R \trianglelefteq G$. ☐

If either $Q \trianglelefteq G$ or $R \trianglelefteq G$, then $QR = RQ \leq G$, and $|QR| = \frac{|Q||R|}{|Q \cap R|} = \frac{|Q||R|}{1} = |Q||R| = qr$. Also, $[G : QR] = |G|/|QR| = pqr/qr = p$ is the smallest prime dividing $|G|$, so $QR \trianglelefteq G$.

- If $Q \trianglelefteq G$, then $Q \trianglelefteq QR$ and so we have the following normal tower:

  $$1 \trianglelefteq Q \trianglelefteq QR \trianglelefteq G,$$

  where the quotients $G/QR \cong P$, $QR/Q \cong R$, and $Q/1 \cong Q$ are abelian because they have prime order.

- If $R \trianglelefteq G$, then $R \trianglelefteq QR$ and so we have the following normal tower:

  $$1 \trianglelefteq R \trianglelefteq QR \trianglelefteq G,$$

  whose quotients are isomorphic to P, Q and R which, again, are abelian.

In both cases, G is solvable. ☐

Problem 4.2.5

Let K be the subgroup of $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ generated by the three elements: $u_1 = (1, -3, -2)$, $u_2 = (1, 3, 2)$, and $u_3 = (3, 3, 4)$. Determine the structure of the quotient G/K as a direct sum of cyclic groups.

Notes and Comments

Proof. Let $i : K \to G$ be the inclusion map of K into G. Then the matrix representation of i, using the basis $\{u_1, u_2, u_3\}$ of K and $\{e_1, e_2, e_3\}$ of G, we have

$$[i] = \begin{pmatrix} 1 & 1 & 3 \\ -3 & 3 & 3 \\ -2 & 2 & 4 \end{pmatrix}.$$

Now, to determine the structure of G/K, we can find the invariant factors of K by computing the Smith Normal Form of $[i]$.[6] Without comment to the exact row and column operations used, we obtain

$$[i] \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 6 & 12 \\ 0 & 4 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 12 \\ 0 & 4 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Thus $i(K) \cong \mathbb{Z} \oplus 2\mathbb{Z} \oplus 6\mathbb{Z}$ and so

$$G/K \cong \frac{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z} \oplus 2\mathbb{Z} \oplus 6\mathbb{Z}} \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z})$$

by the Chinese Remainder Theorem. □

Problem 4.2.6

Let R be a commutative ring. An R-module M is *flat* if the function $M \otimes_R (\cdot)$ is exact. Prove that any projective R-module is flat.

Notes and Comments

Proof. Recall that $M \otimes_R (\cdot)$ is always right exact. Thus it is enough to show that $M \otimes_R (\cdot)$ is left exact. That is, if $0 \longrightarrow N' \xrightarrow{\psi} N$ is exact, we want to show that

$$0 \longrightarrow M \otimes_R N' \xrightarrow{1\otimes\psi} M \otimes_R N$$

is also exact. That is, $1 \otimes \psi$ is injective.

Case 1: First we will prove that any free module F is flat. Let S be a basis for F. Choose $x \in \ker 1\otimes\psi$. Since F is free, x can be uniquely written as $\boxed{x = \sum_{s\in S} s \otimes n'_s}$. Now we have

$$0 = (1 \otimes \psi)(x) = \sum_{s\in S} s \otimes \psi(n'_s).$$

Hence for all $s \in S$, we have $\psi(m'_s) = 0$. Since ψ is injective, this means $m'_s = 0$ for all $s \in S$. Thus $x = 0$ and so $1 \otimes \psi$ is injective. Hence F is flat.

Case 2: Now suppose that P is projective. Then, equivalently, P is the direct summand of a free module F and so $F = P \oplus Q$. Then we have an exact sequence

$$0 \longrightarrow F \otimes_R N' \xrightarrow{1\otimes\psi} F \otimes_R N \quad .$$

Let $\varphi : P \otimes_R N' \to P \otimes_R N$ denote the map induced by ψ.[7] Now we have isomorphisms

$$F \otimes_R N' \cong (P \otimes_R N') \oplus (Q \otimes_R N') \text{ and } F \otimes_R N \cong (P \otimes_R N) \oplus (Q \otimes_R N).$$

[6] For those unfamiliar with Smith Normal Form, it's a very useful tool and worthwhile to learn.
[7] It was tempting to call this map $1 \otimes \psi$ in quotes.

Thus we have a commutative diagram

$$\begin{tikzcd}
F \otimes_R N' \arrow[r, hook, "1\otimes\psi"] & F \otimes_R N \arrow[d, "\pi"] \\
P \otimes_R N' \arrow[u, hook] \arrow[r, "\varphi"] & P \otimes_R N
\end{tikzcd}.$$

Hence φ is an injective map as well. Thus $P \otimes_R (\cdot)$ is exact. That is, P is flat. $\square$

Algebra Exam

Summer 2013

Problem 4.3.1

Let P be a p-Sylow subgroup of a finite group G such that for every other p-Sylow subgroup Q we have $P \cap Q = \{1\}$. Show that any pair P_1, P_2 of p-Sylow subgroups intersects trivially: $P_1 \cap P_2 = \{1\}$.

Notes and Comments

Proof. We will show that all $p-$Sylow subgroups in G intersect trivially. By the second Sylow theorem, all p-Sylow subgroups are conjugate in G. Let P_1 and P_2 be two distinct p-Sylow subgroups of G and $h \in G$ such that $hP_1h^{-1} = P$. Notice that, since P_1 and P_2 are distinct, $hP_2h^{-1} \neq P$.

Let $Q = P_1 \cap P_2$ and consider conjugating by h. For any $q \in Q$, we have $hqh^{-1} \in hP_1h^{-1} = P$ and also $hqh^{-1} \in hP_2h^{-1} \neq P$. Thus $q \in P \cap hP_2h^{-1}$ and so $q = 1$ by the hypothesis on P. Hence $P_1 \cap P_2 = \{1\}$. $\square$

Problem 4.3.2

Let k be a field and x, y indeterminates over k.

(a) Show that x and y are irreducible in $k[x, y]$.

(b) Show that, as rings, $k[x, y]/(y - x^2)$ can never be isomorphic to $k[x, y]/(y^2 - x^2)$.

(c) Determine the structure of the quotient ring $\mathbb{Q}[x]/(x^{12} - 1)$ by characterizing this ring as a direct product of simple (quotient) rings.

Notes and Comments

Proof of (a). We will show that x and y are irreducible in $k[x, y]$. Since k is a field, we have that $k[x]$ is a PID and hence $k[x, y]$ is a UFD. Thus irreducibles are prime.[8] Thus it suffices to show that x and y are prime. Furthermore, this is equivalent to showing that $\langle x \rangle$ and $\langle y \rangle$ are prime ideals. However, we know that $k[x, y]/\langle x \rangle \cong k[y]$ which is entire.[9] Thus $\langle x \rangle$ is a prime ideal and x is irreducible as desired. By a symmetric argument, y is also irreducible. $\square$

Proof of (b). We will show that $k[x, y]/\langle y - x^2 \rangle \ncong k[x, y]/\langle x^2 - y^2 \rangle$. Notice that, in $k[x, y]/\langle x^2 - y^2 \rangle$, we have that $(x + y)(x - y) = x^2 - y^2 = 0$ but neither $x + y \equiv 0$ or $x - y \equiv 0 \pmod{x^2 - y^2}$. Hence $\boxed{k[x, y]/\langle x^2 - y^2 \rangle \text{ is not entire}}$. On the other hand, we claim that $k[x, y]/\langle y - x^2 \rangle \cong k[x]$.

Consider the diagram

$$\begin{tikzcd}
k[x,y] \arrow[r, "\varphi"] \arrow[d, two heads, "\pi"'] & k[x] \\
k[x,y]/\langle y - x^2 \rangle \arrow[ur, dotted, "\overline{\varphi}"'] &
\end{tikzcd}$$

where $\varphi : k[x, y] \to k[x]$ acts by $\varphi(f(x, y)) = f(x, x^2)$. Clearly, φ is surjective since, for any $g \in k[x]$, we have $\varphi(g(x)) = g(x)$. Additionally, we have $\langle y - x^2 \rangle \subseteq \ker(\varphi)$, so it remains to show the opposite inclusion.

[8] Moreover, as $k[x, y]$ is entire (i.e., an integral domain in Lang terminology), every prime is irreducible.

[9] Actually, it's a PID but we don't need all of that structure.

Let $h(x,y) \in \ker(\varphi)$. Viewing h as a polynomial in $k[x][y]$, we notice that $y - x^2$ is a monic degree one polynomial. By the Division Algorithm, we can write $h = (y-x^2)q(x,y) + r(x,y)$ with y-degree of r equal to zero. That is, r is just a polynomial in x.

Since $h \in \ker(\varphi)$, we have

$$0 = h(x, x^2) = (x^2 - x^2)q(x, x^2) + r(x) = r(x).$$

Thus $h = (y-x^2)q$ and so $h \in \langle y - x^2 \rangle$ as desired. Therefore, $\ker(\varphi) = \langle y - x^2 \rangle$. Hence $\overline{\varphi}$ is an isomorphism and so $\boxed{k[x,y]/\langle y - x^2\rangle \text{ is a PID}}$ and hence has no zero divisors, unlike $k[x,y]/\langle y^2 - x^2\rangle$.[10] □

Proof of (c). We want to characterize $\mathbb{Q}[x]/\langle x^{12} - 1\rangle$ as a product of simple quotient rings. From the theory of cyclotomic polynomials, we know that

$$x^{12} - 1 = \prod_{d|12} \Phi_d = (x-1)(x+1)(x^2+x+1)(x^2+1)(x^2-x+1)(x^4-x^2+1)$$

where each factor is irreducible over $\mathbb{Q}$. We know that $\mathbb{Q}$ is a PID and hence the ideals generated by each divisor is maximal. These principal ideals are all distinct and hence these ideals are pairwise comaximal. Thus, by the Chinese Remainder Theorem, we have

$$\mathbb{Q}[x]/\langle x^{12} - 1\rangle = \prod_{d|12} \mathbb{Q}[x]/\langle \Phi_d \rangle$$

as desired. □

Problem 4.3.3

Let V be a finite–dimensional vector space over a field k, and let $T : V \to V$ be a linear operator whose characteristic polynomial generates the ideal $I \subseteq k[x]$ in the polynomial ring consisting of polynomials that vanish at T, i.e., $I = \{f \in k[x] : f(T) = 0\}$. Show that any linear operator $U \in \mathrm{End}_k(V)$ that commutes with T is a polynomial in T, i.e., if $UT = TU$, then there is some $p \in k[x]$ such that $U = p(T)$.

Notes and Comments

Proof. Let V be a finite dimensional vector space over k and $T \in \mathrm{End}_k(V)$ such that the minimal polynomial of T is equal to its characteristic polynomial. Observe that this is equivalent to the condition given in the problem statement. We will show that any $U \in \mathrm{End}_k(V)$ that commutes with T is a polynomial in T. Since the minimal polynomial of T is equal to its characteristic polynomial, we that V decomposes as a cyclic $k[x]$-module with action $p \cdot v = p(T)v$. Thus there exists a vector $x \in V$ such that we can form a basis for V of the form $\{x, Tx, T^2x, \ldots, T^{n-1}x\}$.

Let U be any operator that commutes with T and express $Ux = \sum_{i=0}^{n-1} u_i T^i x$. Define $p \in k[x]$ by $\sum_{i=0}^{n-1} u_i x^i$. We claim that $U = p(T)$.

[10] Note this is the same as showing that $x^2 - y^2$ is reducible but $y - x^2$ is not.

Letting $v \in V$ be arbitrary, we may express v in terms of our basis as $v = \sum_{j=0}^{n-1} v_j T^j x$. Then, since U commutes with T, we have

$$\begin{aligned} U(v) = U\left(\sum_{j=0}^{n-1} v_j T^j x\right) = \sum_{j=0}^{n-1} v_j T^j U x &= \sum_{j=0}^{n-1} v_j T^j \sum_{i=0}^{n-1} u_i T^i x \\ &= \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} u_i v_j T^{j+i} x \\ &= \sum_{i=0}^{n-1} u_i T^i \sum_{j=0}^{n-1} v_j T^j x = p(T) \cdot \sum_{j=0}^{n-1} v_j T^j x = p(T) \cdot v. \end{aligned}$$

Thus $U = p(T)$ as desired. □

Problem 4.3.4

Let E, F, and K be fields all contained in some larger extension Ω.

(a) Suppose the $K \subset F \subset E$. Show that E/F and F/K are algebraic extensions implies that E/K is also algebraic.

(b) Suppose that E/K is an algebraic extension, but that F/K is an arbitrary extension. Show that the extension EF/F is algebraic where EF is the compositum of E and F.

Notes and Comments

Proof of (a). Let $e \in E$ be arbitrary. Since E/F is algebraic, e has a minimal polynomial in $F[x]$ with coefficients $f_0, f_1, \ldots, f_n$. Then $K(f_0, f_1, \ldots, f_n, e)/K(f_0, f_1, \ldots, f_n)$ is a finite algebraic extension. Similarly, since F/K is algebraic, we have that each f_i is algebraic over K and hence that $K(f_0, f_1, \ldots, f_n)/K$ is a finite algebraic extension. By multiplicativity of degrees in towers, $K(f_0, f_1, \ldots, f_n, e)/K$ is finite and hence algebraic. Since e was arbitrary, every element of E is algebraic over K. □

Proof of (b). Let $L \subseteq EF$ be the set of elements of EF that are algebraic over F. Recall that this is a field since the algebraic elements are closed under the field operations. We definitely have $F \subset L$ since every element of F satisfies an obvious linear polynomial in $F[x]$. Similarly, $K \subseteq F$ and E/K is algebraic, so $E \subset L$. As L is a field, it must thus contain the compositum EF. Hence $L = EF$. □

Problem 4.3.5

Let k be a field and let V and W be k-vector spaces. Let $V^* := \mathrm{Hom}_k(V, k)$ denote the dual space of V.

(a) Define a natural map $F : V^* \otimes W \to \mathrm{Hom}_k(V, w)$ of vector spaces that is an isomorphism if V and W are finite-dimensional. (Be sure to show that F is well-defined. You need not prove naturality, but be sure to state what it means that F is natural.)

(b) Recall that a *projection* **on a finite-dimensional k-vector space V is an idempotent linear operator $P \in \mathrm{End}_k(V)$. Determine necessary and sufficient conditions on $\varphi \in V^*$ and $v \in V$ insuring that the decomposable tensor $\varphi \otimes v \in V^* \otimes V$ corresponds, via the linear isomorphism $F : V^* \otimes V \to \mathrm{End}(V)$ above, to a nonzero projection operator.**

Notes and Comments

Proof of (a). We will show that $V^* \otimes W$ is naturally isomorphic to $\mathrm{Hom}(V, W)$. We will construct a isomorphism out of the tensor product by defining a (clearly) bilinear map $\overline{f} : V^* \times W \to \mathrm{Hom}(V, W)$ by $\overline{f}(\varphi, w)(v) = \varphi(v)w$. By the universal mapping property of the tensor product, this induces a well-defined map $f : V^* \otimes W$ such that the obvious diagram commutes.

$$\begin{tikzcd}
V^* \otimes W \arrow[r, dotted, "f"] & \mathrm{Hom}_K(V, W) \\
V \times W \arrow[u, hook, "i"] \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

To see that this is a natural map, we need to work in the category of k-vector spaces with k-linear arrows. Our two functors are $F_1 : W \to V^* \otimes W$ and $F_2 : W \to \mathrm{Hom}(V, W)$. Let A, B be k-vector spaces and $T : A \to B$ a k-linear map. These functors act on maps[11] by $F_1(T)(\varphi \otimes a) = \varphi \otimes T(a)$ and $F_2(T)(\varphi(\cdot)a) = \varphi(\cdot)T(a)$. Then the following diagram commutes and our map $\overline{f}$ is natural:

$$\begin{tikzcd}
\varphi \otimes a \arrow[rrr, mapsto, bend left] \arrow[ddd, mapsto, bend right] & & & \varphi(\cdot)a \arrow[ddd, mapsto, bend left] \\
& V^* \otimes A \arrow[r, "\varphi"] \arrow[d, "F_1(f)"'] & \mathrm{Hom}(V, A) \arrow[d, "F_2(f)"] & \\
& V^* \otimes B \arrow[r, "\varphi"'] & \mathrm{Hom}(V, B) & \\
\varphi \otimes T(a) \arrow[rrr, mapsto, bend right] & & & \varphi(\cdot)T(a)
\end{tikzcd}$$

If V and W are finite-dimensional then, considering $\{\psi_1, \ldots, \psi_n\}$ and $\{e_1, \ldots, e_m\}$ the standard bases for V^* and W respectively, f takes a basis to a basis ($f(\psi_i \otimes e_j)(v) = \psi_i(v)e_j$) and is hence an isomorphism. $\square$

[11] And sums of elementary tensors by linear extensions of this.

Proof of (b). To determine necessary and sufficient conditions for $\varphi \otimes v \in V^* \otimes V$ to be mapped to a non-zero projection operator by f, as given in part (a), note that

$$f(\varphi \otimes v)^2(w) = f(\varphi \otimes v)(w) \Leftrightarrow \varphi(\varphi(w)v)v = \varphi(w)v \Leftrightarrow \varphi(w)\varphi(v)v = \varphi(w)v$$

Certainly, in order for $f(\varphi \otimes v)$ to be non-zero we need that $\varphi \neq 0$ and $v \neq 0$. Additionally, from the above, we must have $\varphi(v) = 1$. This condition is necessary and sufficient since, for any non-zero $\psi \in V^*$, any element $x \in V$ that has $\psi(x) = 1$ makes $f(\psi \otimes x)$ a projection operator by the computation above. Similarly, for any $y \in V$ and $\nu \in V^*$ such that $\nu(y) = 1$, we also have that $f(\nu \otimes y)$ is a projection operator. □

Problem 4.3.6

Let K/F be a finite separable extension and L the Galois closure of K in some algebraic closure $\overline{F}$ of F. Let G be the Galois group $\mathrm{Gal}(L/F)$ and H the subgroups corresponding to K under the Galois correspondence.

(a) Show that there is a one-to-one correspondence between the set of embeddings $\sigma : K/F \to \overline{F}$ (that is, of K into $\overline{F}$ fixing F pointwise) and all the cosets of G/H.

(b) Recall that one defines the norm from K to F as follows: For $\alpha \in K$, define $N_{K/F}(\alpha) = \prod_\sigma \sigma(\alpha)$ where the product is taken over all embeddings $\sigma : K/F \to \overline{F}$. Show that $N_{K/F}(K) \subseteq F$.

Notes and Comments

Proof of (a). Since L is normal over F, we have that L is normal over K.[12] Thus every embedding $\sigma : K/F \to \overline{F}$ lifts to $[L : K]$ automorphisms of L that fixes K. In particular, any such embedding $\overline{\sigma}$ is an element of G since $K \subseteq F$. Also, for any $h \in H$, we also have that $\overline{\sigma}h$ still lies above σ.

Finally, we note that for any distinct $\sigma, \tau : K/F \to \overline{F}$ and $h \in H$, we cannot have $\overline{\sigma}h|_K = \tau$ since σ and τ must differ on some element of K which is fixed by h. Thus, the map that takes σ to $[\overline{\sigma}]$ is an bijection. □

Proof of (b). Let $\alpha \in K$ be selected arbitrarily. Then, by definition, $N_{K/F}(\alpha) = \prod\limits_{\sigma : K/F \to \overline{F}} \sigma(\alpha)$. For any element $g \in Gal(L/F)$, we have that $g(N_{K/F}(\alpha)) = N_{K/F}(\alpha)$ since g simply permutes the σ. Thus $N_{K/F}(\alpha)$ is fixed by the entire Galois group and must lie in the base field F. □

[12] Which is why H is a (sub-)group.

Algebra Exam

Fall 2013

Problem 4.4.1

Let V be a 3-dimensional $\mathbb{Q}$-vector space, and let $T : V \to V$ be a linear operator that has eigenvalues 1 and 2 but is not diagonalizable.

(a) What are the possible rational canonical forms of T?.

(b) What are the possible Jordan canonical forms of the operator $\mathrm{Id} \otimes T : \mathbb{C} \otimes_{\mathbb{Q}} V \to \mathbb{C} \otimes_{\mathbb{Q}} V$ on the complexification?

Notes and Comments

Proofs of (a) and (b). First note that, since V has dimension three, the degree of the characteristic polynomial of T, χ_T, is 3 as well. Furthermore, the roots of χ_T are eigenvalues of T and, since T is not diagonalizable, it cannot have 3 distinct eigenvalues. So χ_T is either $(X-1)^2(X-2)$ or $(X-1)(X-2)^2$.

Now, since the minimal polynomial μ_T divides χ_T (and they share the same set of roots), we consider four possible cases:

1. $\chi_T = (X-1)^2(X-2)$ and $\mu_T = (X-1)(X-2)$
2. $\chi_T = (X-1)^2(X-2)$ and $\mu_T = (X-1)^2(X-2)$
3. $\chi_T = (X-1)(X-2)^2$ and $\mu_T = (X-1)(X-2)$
4. $\chi_T = (X-1)(X-2)^2$ and $\mu_T = (X-1)(X-2)^2$

Each of these cases gives rise to a unique rational canonical form, some of which may be diagonalizable. We will defer identifying the diagonalizable forms until later. In all cases below, we determine the invariant factors of V viewed as $\mathbb{Q}[X]$-module via T.

- **Case 1:** $\chi_T = (X-1)^2(X-2)$ and $\mu_T = (X-1)(X-2)$.

  Then the invariant factors of V are $(X-1)$ and $(X-1)(X-2)$. Thus $V \cong \frac{\mathbb{Q}[X]}{(X-1)} \oplus \frac{\mathbb{Q}[X]}{(X-1)(X-2)}$ and the associated companion matrices are

  $$C((X-1)) = \begin{bmatrix} 1 \end{bmatrix} \text{ and } C((X-1)(X-2)) = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$$

  since $(X-1)(X-2) = X^2 - 3X + 2$. Hence the rational canonical form is given by $R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$.

- **Case 2:** $\chi_T = (X-1)^2(X-2)$ and $\mu_T = (X-1)^2(X-2)$.

  Here the single invariant factor is $(X-1)^2(X-2)$. Thus $V \cong \frac{\mathbb{Q}[X]}{(X-1)^2(X-2)}$ and the associated companion matrix is also the rational canonical form $R_2 = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -5 \\ 0 & 1 & 4 \end{bmatrix}$ since $(X-1)^2(X-2) = X^3 - 4X^2 + 5X - 2$.

- **Case 3:** $\chi_T = (X-1)(X-2)^2$ and $\mu_T = (X-1)(X-2)$.

  Here the invariant factors of V are $(X-2)$ and $(X-1)(X-2)$. Thus $V \cong \frac{\mathbb{Q}[X]}{(X-2)} \oplus \frac{\mathbb{Q}[X]}{(X-1)(X-2)}$. Then the associated companion matrices are:

  $$C((X-2)) = \begin{bmatrix} 2 \end{bmatrix} \text{ and } C((X-1)(X-2)) = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}.$$

  Thus the rational canonical form is given by $R_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$.

- **Case 4:** $\chi_T = (X-1)(X-2)^2$ and $\mu_T = (X-1)(X-2)^2$.

  Here the invariant factor of V is just $(X-1)(X-2)^2$. Thus $V \cong \frac{\mathbb{Q}[X]}{(X-1)(X-2)^2}$ and the rational canonical form is given by $R_4 = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & -8 \\ 0 & 1 & 5 \end{bmatrix}$ since $(X-1)(X-2)^2 = X^3 - 5X^2 + 8X - 4$.

To determine which of the above forms are diagonalizable, we consider the corresponding Jordan canonical forms.[13] Note that Jordan canonical forms are defined in general only over algebraically closed fields. Since $\mathbb{Q}$ is not algebraically closed, we must consider the complexification, as mentioned in part (b). However, since all roots of χ_T (the eigenvalues 1 and 2) do lie in $\mathbb{Q}$, we can proceed as we normally would in finding the Jordan forms.

- **Case 1:** We previously showed that the invariant factor decomposition of V was $\frac{\mathbb{Q}[X]}{(X-1)} \oplus \frac{\mathbb{Q}[X]}{(X-1)(X-2)}$. Utilizing the Chinese Remainder Theorem, we can convert this to the elementary divisor decomposition of V: $V \cong \frac{\mathbb{Q}[X]}{(X-1)} \oplus \frac{\mathbb{Q}[X]}{(X-1)} \oplus \frac{\mathbb{Q}[X]}{(X-2)}$. Thus the Jordan blocks are $\begin{bmatrix} 1 \end{bmatrix}$, $\begin{bmatrix} 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \end{bmatrix}$ and so the Jordan canonical form is $J_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Since this is diagonal, this is not a possible form for T. Furthermore, this proves that the corresponding rational canonical form R_1 is not possible either.

- **Case 2:** The invariant factor decomposition here was $V \cong \frac{\mathbb{Q}[X]}{(X-1)^2(X-2)}$, so we get an elementary divisor decomposition of V to be $V \cong \frac{\mathbb{Q}[X]}{(X-1)^2} \oplus \frac{\mathbb{Q}[X]}{(X-2)}$. Our Jordan blocks are $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \end{bmatrix}$. Thus the Jordan form is $J_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Since this is not diagonal, this is a possible Jordan form and the corresponding rational canonical form R_2 is also valid.

- **Case 3:** Here we obtain an elementary divisor decomposition of V as $\frac{\mathbb{Q}[X]}{(X-2)} \oplus \frac{\mathbb{Q}[X]}{(X-2)} \oplus \frac{\mathbb{Q}[X]}{(X-2)}$. However, since these all lead to single Jordan blocks, this will also lead to a diagonal Jordan form and thus is not possible.

[13] Since any diagonalizable matrix has a diagonal Jordan form and vice versa.

- **Case 4:** Here we obtain the following elementary divisor decomposition: $V \cong \frac{\mathbb{Q}[X]}{(X-1)} \oplus \frac{\mathbb{Q}[X]}{(X-2)^2}$. This gives rise to the Jordan blocks $\begin{bmatrix} 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and the Jordan canonical form $J_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

In summary, the possible rational canonical forms for T are

$$R_{T,1} = R_2 = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -5 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } R_{T,2} = R_4 = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & -8 \\ 0 & 1 & 5 \end{bmatrix}$$

and the possible Jordan canonical forms for T are

$$J_{T,1} = J_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } J_{T,2} = J_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

and no others exist. □

Problem 4.4.2

Let A be an integral domain.

(a) Define what it means for an element $\pi \in A$ to be irreducible.

(b) Suppose that $\pi \in A$ is irreducible. Show that the polynomial ring $A[x]$ is not a PID.

(c) Show that $A[x]$ is a PID if and only if A is a field.

Notes and Comments

Proof of (a). A non-unit element $\pi \in A$ is *irreducible* if π is not the product of two non-units. Equivalently, if $\pi = ab$ implies that either a or b is a unit. □

Proof of (b). We will prove that the ideal $(\pi, x) \subseteq A[x]$ is not principal. Suppose for contradiction that $(\pi, x) = (f)$. Then $\pi \in (f)$, so f divides π. Thus $f \in A$ by degree considerations. Since π is irreducible, either f is a unit or f is an associate of π. The latter case is impossible because $x \notin (\pi)$ (if it were, π would be a unit). So f is a unit.

Since $f \in (\pi, x)$, we can write $f = \pi g + xh$ for some $g, h \in A[x]$. Looking at the constant term of this expression, we obtain $f = \pi g_0$ where $g_0 \in A$ is the constant term of g. However, since f is a unit, so is π. $\lightning$ Thus $A[x]$ is not a PID. □

Proof of (c). ($\Rightarrow$): Assume $A[x]$ is a PID. To obtain a contradiction, suppose $c \in A$ with $c \neq 0$ is not a unit in A. Since A is an integral domain, c is also not a unit in $A[x]$. So $(c) \subsetneq A[x]$ and so $A[x]$ has a maximal ideal (p) that contains (c).[14] Then $c \in (p)$, so p divides c and thus $p \in A$.

We will prove that p is an irreducible element of A. Suppose $p = ab$ for $a, b \in A$. Since $a, b \in A[x]$ too, we have $p \in (a)$ (the ideal in $A[x]$). Thus $(p) \subseteq (a)$. But (p) is maximal, so either $(a) = (p)$ or $(a) = A[x]$.

[14] Since $A[x]$ is a PID, this ideal is necessarily principal.

- If $(a) = (p)$, then a and p are associate. Thus $p = au$ for a unit $u \in A[x]$. Then $au = p = ab$, so $u = b$. Hence b is a unit.

- If $(a) = A[x]$, then a is a unit.

Therefore, p is irreducible in A. By part (b), $A[x]$ is not a PID. $\lightning$ Therefore every non-zero $c \in A$ is a unit, i.e., A is a field.

($\Leftarrow$): Assume A is a field and let I be an ideal in $A[x]$. If $I = (0)$, then I is principal. If $I \neq (0)$, then I has non-zero elements. Let $f \in I$ be a non-zero element with minimum degree. We will show that $(f) = I$.

We know $(f) \subseteq I$ because $f \in I$. Now let $g \in I$. Since A is a field and $f \neq 0$, we can use the Division Algorithm to obtain $q, r \in A[x]$ such that $g = qf + r$ and $\deg r < \deg f$. Thus $r = g - qf \in I$ (because $f, g \in I$). But f has minimum degree among non-zero elements of I, so $r = 0$. Thus $g = qf$ and so $g \in (f)$. Therefore $(f) = I$ and I is principal. Hence $A[x]$ is a PID. $\square$

Problem 4.4.3

Let V be a finite-dimensional vector space over a field k of characteristic zero and let $\langle\cdot,\cdot\rangle : V \times V \to k$ be a skew-symmetric bilinear form.

(a) State what it means to say that the form is *non-degenerate*.

(b) Let $W \subseteq V$ be a subspace such that the restriction $\langle\cdot,\cdot\rangle|_{W\times W} : W \times W \to k$ is non-degenerate. Show that V admits an orthogonal decomposition $V = W \boxplus W^\perp$, where $W^\perp = \{x \in V : \forall w \in V, \langle x, w\rangle = 0\}$. Show also that if the bilinear form on V was non-degenerate, then so is the restriction to $W^\perp$.

(c) Show that if the form is non-degenerate on V, then V is even-dimensional and it has a basis relative to which the Gram matrix of the form is

$$\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix.

Notes and Comments

Proof of (a). The form $\langle\cdot,\cdot\rangle$ is non-degenerate if, when viewed as a linear map $B : V \to V^*$ given by $B(v)(w) = \langle v, w\rangle$, B is an isomorphism. Equivalently, for finite-dimensional vector spaces, $\forall x \in V$ with $x \neq 0$, $\exists w \in V$ such that $\langle x, w\rangle \neq 0$. $\square$

Proof of (b). $\underline{W \cap W^\perp = \{0\}}$: Let $x \in W \cap W^\perp$. If $x \neq 0$ then, since $x \in W$ and $\langle\cdot,\cdot\rangle|_{W\times W}$ is nondegerate, $\exists w \in W$ such that $\boxed{\langle x, w\rangle \neq 0}$. However, since $x \in W^\perp$, we must have $\boxed{\langle x, w\rangle = 0}$. $\lightning$ Hence $x = 0$ and so $W \cap W^\perp$ is trivial.

$\underline{V = W + W^\perp}$: Consider $\langle\cdot,\cdot\rangle|_{V\times W}$ as a linear map $B : V \to W^*$. By the Rank-Nullity Theorem, we know that

$$\dim V = \dim\ker B + \dim\operatorname{im} B.$$

Notice that, by definition,

$$\ker B = \{x \in V \mid B(v)(w) = 0\ \forall w \in W\} = \{x \in V \mid \langle x, w\rangle = 0\ \forall w \in W\} = W^\perp.$$

On the other hand, $B|_W : W \to W^*$ is an isomorphism since $\langle\cdot,\cdot\rangle|_{W\times W}$ is non-degenerate. Hence $\operatorname{im} B \cong W^* \cong W$.

Thus

$$\dim V = \dim W^\perp + \dim W.$$

Since $W \cap W^\perp$ is trivial, $\boxed{V = W \boxplus W^\perp}$ as desired.

Finally, assume $\langle\cdot,\cdot\rangle$ is non-degenerate on V. Let $0 \neq x \in W^\perp$. Since $\langle\cdot,\cdot\rangle$ is non-degenerate, there is some $v \in V$ such that $\langle x, v\rangle \neq 0$. By our work above, $v = w + w^\perp$ for some $w \in W$, $w^\perp \in W^\perp$. By linearity and the definition of $W^\perp$, we now have

$$0 \neq \langle x, v\rangle = \langle x, w + w^\perp\rangle = \langle x, w\rangle + \langle x, w^\perp\rangle = \langle x, w^\perp\rangle.$$

That is, $\boxed{\langle x, w^\perp\rangle \neq 0}$ and so $\langle\cdot,\cdot\rangle|_{W^\perp\times W^\perp}$ is non-degenerate as desired. □

Proof of (c). Since $\langle\cdot,\cdot\rangle$ is skew-symmetric, any Gram matrix G of $\langle\cdot,\cdot\rangle$ will be skew-symmetric. Hence

$$\det G = \det G^T = \det(-G) = (-1)^{\dim V} \det G.$$

Since $\langle\cdot,\cdot\rangle$ is non-degenerate, $\det G \neq 0$. Hence $1 = (-1)^{\dim V}$. As the characteristic of k is not 2, this means that $\boxed{\dim V \text{ is even}}$.

To find the desired basis, we will proceed by induction on n where $\dim V = 2n$.

If $n = 1$, let $v \neq 0$ be in V. By nondegeneracy, $\exists w \in V$ such that $\langle w, v\rangle \neq 0$. We may further assume that $\langle w, v\rangle = 1$ because, otherwise, we could rescale w.

Let $W = \operatorname{span}(v, w)$. Then $\langle\cdot,\cdot\rangle|_{W\times W}$ is nondegerate and has Gram matrix (with respect to this basis) $G = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

For the inductive step, assume $\dim V = 2n + 2$. Our inductive hypothesis states that, for any $\leq 2n$-dimensional vector space with a non-degenerate skew-symmetric bilinear form, we have the desired basis.

Define W as above. By part (b), $V = W \boxplus W^\perp$ and $\langle\cdot,\cdot\rangle|_{W^\perp\times W^\perp}$ is a non-degenerate (necessarily skew-symmetric) bilinear form. Thus, by the inductive hypothesis and the base case, $W^\perp$ and W have bases $\{v_1, \ldots, v_n, w_1, w_n\}$ and $\{v, w\}$ with the desired form for the respective forms.

Consider the basis $\{v, v_1, \ldots, v_n, w, w_1, \ldots, w_n\}$ for V. In this basis, the Gram matrix of $\langle\cdot,\cdot\rangle$ is indeed

$$\begin{bmatrix} 0 & -I_{n+1} \\ I_{n+1} & 0 \end{bmatrix}$$

by the definition of $W^\perp$ (thus giving us the zeros we might worry about). □

Problem 4.4.4

Let K be a field of prime characteristic p, $\mathbb{F}_p$ the finite field with p elements.

(a) First assume that $K/\mathbb{F}_p$ is an algebraic extension. Show that, for every $\alpha \in K$, there is a unique $\beta \in K$ with $\beta^p = \alpha$.

(b) Now let K be an arbitrary field of characteristic p and assume that L/K is a finite extension with $[L : K] = n$ and $p \nmid n$. Show that L/K is a separable extension of fields.

Notes and Comments

Proof of (a). Let $\alpha \in K$. Since K is algebraic over $\mathbb{F}_p$, there is an irreducible polynomial $f(x) = \sum_{i=0}^{n} c_i x^i$ where $f \in \mathbb{F}_p[x]$ such that $f(\alpha) = 0$.

Consider the field $\mathbb{F}_p(\alpha)$. Since this is a finite extension of $\mathbb{F}_p$, it is necessarily a finite field. Hence the Frobenius map is an automorphism. Thus $\alpha = \beta^p$ for some unique $\beta \in \mathbb{F}_p(\alpha)$. As β is contained in a subextension of $K/\mathbb{F}_p$, we know that $\boxed{\beta \in K}$.

To show uniqueness of β, suppose $\exists \gamma \in K$ so that $\gamma^p = \alpha$. Then, since $\mathbb{F}_p$ is a finite field, the Frobenius map is again an automorphism and hence each coefficient of f can be written uniquely as $c_i = b_i^p$ for some $b_i \in \mathbb{F}_p$. Hence, as K has characteristic p,

$$0 = f(\alpha) = f(\beta^p) = \sum_{i=0}^{n} c_i(\beta^p)^i = \sum_{i=0}^{n} b_i^p(\beta^i)^p = \sum_{i=0}^{n}(b_i\beta^i)^p = \left(\sum_{i=0}^{n} b_i\beta^i\right)^p.$$

As K is a field, this means that

$$\sum_{i=0}^{n} b_i\beta^i = 0.$$

Suppose $\gamma^p = \alpha = \beta^p$. Then γ also satisfies $\sum_{i=0}^{n} b_i\gamma^i = 0$. Since $\mathbb{F}_p(\alpha)$ is a finite field, it is a normal extension of $\mathbb{F}_p$ and so must contain every root of this polynomial. That is, $\gamma \in \mathbb{F}_p(\alpha)$ as well. However, as the Frobenius map is an automorphism, this means $\gamma = \beta$. Hence $\boxed{\beta \text{ is unique}}$. $\square$

Proof of (b). Let $\alpha \in L$ and $f = \min_{\alpha,K}$ its minimal polynomial. Suppose f is inseparable. Then $f(x) = \sum_{i=0}^{k} c_i x^i$ where $k = \deg f \mid n$. Hence $\boxed{p \nmid k}$. Since f is inseparable, its formal derivative Df must be the zero polynomial. That is,

$$Df(x) = \sum_{i=1}^{k} ic_i x^{i-1} = 0.$$

So $kc_k = 0$. However, since L is a field, this means $k = 0$ or $c_k = 0$. As $p \nmid k$ and $c_k \neq 0$ by assumption, we have a contradiction. That is, f must be separable and hence L/K is a separable extension. $\square$

Problem 4.4.5

A non-abelian group G has exactly three conjugacy classes. What group is G and why?

Notes and Comments

Proof. Recall the Class Equation, $|G| = |Z(G)| + \sum_i \frac{|G|}{|C_G(x_i)|}$, where each x_i is a representative of a different conjugacy class and $C_G(x_i)$ is the centralizer of that element.[15] Each element of the center is its

[15]Recall x and y in the same conjugacy class implies that $C_G(x) = C_G(y)$.

own conjugacy class, so we can really rewrite the class equation as $|G| = \sum_i \frac{|G|}{|C_G(x_i)|}$ where we include singleton conjugacy classes.[16]

In this problem, we know that G has exactly three conjugacy classes. The identity always commutes with every element, so one of our conjugacy classes is guaranteed to be of order 1. The other two we know nothing about so far, except that the class equation must be satisfied. As it turns out, this is all the information we need.

Let's call the centralizers for our other two conjugacy classes H and K. WLOG, we assume that $|H| \leq |K|$. So the Class Equation simplifies to $|G| = 1 + \frac{|G|}{|H|} + \frac{|G|}{|K|}$ or, equivalently,

$$\boxed{1 = \frac{1}{|G|} + \frac{1}{|H|} + \frac{1}{|K|}} (*).$$

Since distinct conjugacy classes are disjoint, we know that $\boxed{|G| \geq 3}$ and thus $\frac{1}{|G|} \leq \frac{1}{3}$. Since $|H| \leq |K| \leq |G|$, we have $\frac{1}{|G|} \leq \frac{1}{|K|} \leq \frac{1}{|H|}$ and so $1 = \frac{1}{|G|} + \frac{1}{|H|} + \frac{1}{|K|} \leq \frac{3}{|H|}$. Thus $\boxed{|H| \leq 3}$. Now, unfortunately, we need to consider cases.

Case 1: Suppose $|H| = 3$. Then

$$1 = \frac{1}{|G|} + \frac{1}{|H|} + \frac{1}{|K|} \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{|K|},$$

so $|K| \leq 3$. As $3 = |H| \leq |K|$, it must be that $|K| = 3$. We can solve for $|G|$ in the Class Equation and get that $|G| = 3$. So G must be $\mathbb{Z}_3$ which does indeed have 3 conjugacy classes, but is abelian and so it's not the desired group.

Case 2: Suppose $|H| = 2$. Since $|K| \leq |G|$, we have $1 = \frac{1}{|G|} + \frac{1}{|H|} + \frac{1}{|K|} \leq \frac{2}{|K|} + \frac{1}{2}$ and so $|K| \leq 4$.

- If $|K| = 4$, then we get $|G| = 4$ by solving $(*)$. As $|H| = 2$, G is nonabelian because the centralizer of some element is a subgroup of order 2 that is not the whole group. But every group of order 4 is abelian, so this is not possible. $\lightning$
- If $|K| = 3$, then we get $|G| = 6$ by solving $(*)$. The only nonabelian group of order 6 is S_3, which does indeed have three conjugacy classes: $\{\mathrm{id}\}, \{r, r^2\}, \{s, rs, r^2s\}$. Here we use the group presentation $\langle r, s | r^3 = 1, s^2 = 1, rsr = s\rangle$. So G could be S_3.
- If $|K| = 2$, then we get $\frac{1}{|G|} = 0$ from $(*)$. Clearly this is no good.

Case 3: Suppose $|H| = 1$. Then we get $1 = \frac{1}{|G|} + 1 + \frac{1}{|K|}$ and so $\frac{1}{|G|} + \frac{1}{|K|} = 0$. This is obviously not possible since group orders must be positive.

Hence our only possible contender was S_3 and so we can conclude that $\boxed{G = S_3}$. $\square$

Problem 4.4.6

Let $n = 13 \cdot 29 = 377$ and $m \geq 3$ a square-free integer. Let L be the splitting field over $\mathbb{Q}$ of $(x^7 - m)(x^n - 1)$.

(a) Determine the splitting field $L/\mathbb{Q}$ and its degree over $\mathbb{Q}$, justifying all steps.

[16] Indeed, for $a \in Z(G)$, $\frac{|G|}{|C_G(a)|} = \frac{|G|}{|G|} = 1$.

(b) Determine whether or not $\mathrm{Gal}(L/\mathbb{Q})$ is abelian.

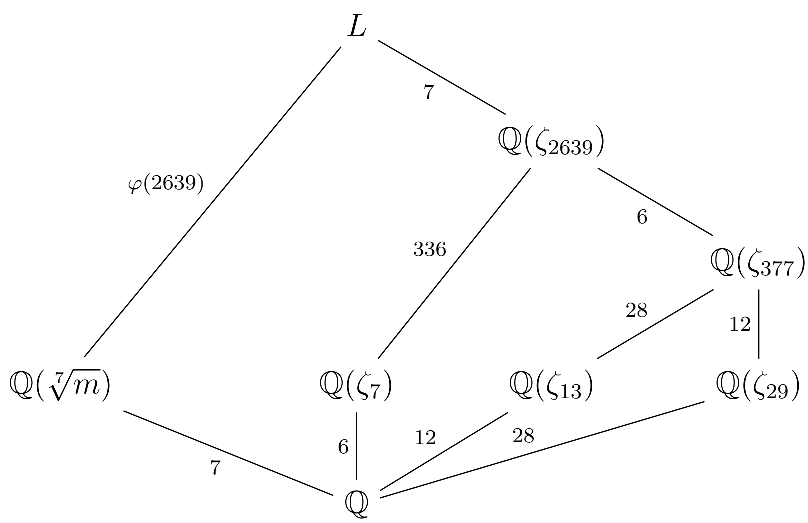
(c) Determine whether or not $\mathrm{Gal}(L/\mathbb{Q})$ is a solvable group and, if so, give an appropriate normal tower which demonstrates this fact. If not, be clear why the extension fails to have a solvable Galois group.

Notes and Comments

Proof of (a). We first note that $x^7 - m$ is irreducible due to Eisenstein's Criterion and its splitting field is generated by $\sqrt[7]{m}$ and ζ_7.[17] On the other hand, we can factor the $x^n - 1$ term using cyclotomic polynomials as

$$x^n - 1 = (x-1)(x^{12} + x^{11} + \cdots + 1)(x^{28} + x^{27} + \cdots + 1)$$

This leads us to the following diagram (note $2639 = 7 \cdot 13 \cdot 29$):



The degree extensions follow from multiplicativity in towers and basics of cyclotomic extensions (intersections and Galois groups). In particular, we have $\sqrt[7]{m} \notin \mathbb{Q}(\zeta_{29})$: $\mathbb{Q}(\sqrt[7]{m})/\mathbb{Q}$ is not Galois hence not a subextension of $\mathbb{Q}(\zeta_{29})/\mathbb{Q}$ (which is abelian and thus every subextension is Galois). This gives $L = \mathbb{Q}(\zeta_7, \zeta_{13}, \zeta_{29}, \sqrt[7]{m}) = \mathbb{Q}(\zeta_{2639}, \sqrt[7]{(m)})$ with degree $14,112 = 7 \cdot (6 \cdot 12 \cdot 28)$ over $\mathbb{Q}$. $\square$

Proof of (b). $\boxed{\text{No}}$ The extension is not abelian because $\mathbb{Q}(\sqrt[7]{m})$ is not normal over $\mathbb{Q}$. If the tower were abelian, all subgroups would be abelian (hence Galois). $\square$

Proof of (c). $\boxed{\text{Yes}}$ The extension is solvable because it is obtained from $\mathbb{Q}$ by adjoining roots of polynomials. Adjoin all of the roots of unity and then the $\sqrt[7]{m}$. $\square$

[17] Look at Lang V1.{3,7,9} to review this material.

Algebra Exam

Summer 2014

Problem 4.5.1

Let R be a commutative ring. An R-module P is *projective* if for all R-module homomorphisms $v : M \to N$ and $f : P \to N$ with v surjective, there exists an R-module homomorphism g lifting f in the sense that the diagram

$$\begin{tikzcd} & M \arrow[d, "v"] \\ P \arrow[ur, dashed, "g"] \arrow[r, "f"'] & N \arrow[d] \\ & 0 \end{tikzcd}$$

commutes. Show that an R-module P is projective if and only if P is a direct summand of a free R-module.

Notes and Comments

Proof. We begin with a helpful lemma.

Lemma 4.5.1 *Every free R-module is projective.*

Proof of Lemma. Let $\mathcal{F}(S)$ be a free R-module on a set S, let M, N be arbitrary R-modules, $f : \mathcal{F}(S) \to N$ an R-module homomorphism, and $v : M \to N$ a surjective R-module homomorphism. Let $\iota : S \to \mathcal{F}(S)$ denote the natural inclusion map.

First we define a set map $\phi : S \to M$. Since v is surjective, there exists $m_s \in M$ such that $v(m_s) = f(\iota(s))$ for all $s \in S$. Define ϕ so that $\phi(s) = m_s$. Since $\mathcal{F}(S)$ is free, there is a unique R-module map $g : \mathcal{F}(S) \to M$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, hook, "\iota"] \arrow[dr, "\phi"'] & \mathcal{F}(S) \arrow[d, dashed, "g"] \\ & M \end{tikzcd}$$

We'll use this to show that

$$\begin{tikzcd} & \mathcal{F}(S) \arrow[dl, "g"'] \arrow[d, "f"] & \\ M \arrow[r, "v"'] & N \arrow[r] & 0 \end{tikzcd}$$

commutes. By construction of ϕ, we have $v \circ \phi = f \circ \iota$. Thus

$$f \circ \iota = v \circ \phi = v \circ (g \circ \iota),$$

where the last equality follows from the commutativity of the previous diagram. So

$$f = v \circ g$$

and so $\mathcal{F}(S)$ is projective. $\square$

Now we will prove the desired result.

($\Longrightarrow$) Assume P is projective. We will show that it is the direct summand of a free R-module. Since every R-module is the quotient of a free R-module, we can write $P \cong \mathcal{F}(S)/Q$, where $\mathcal{F}(S)$ is a free R-module and Q is some R-module.

Let $\pi : \mathcal{F}(S) \to \mathcal{F}(S)/Q$ denote the standard projection and ϕ the isomorphism between $\mathcal{F}(S)/Q$ and P. Since P is projective and π is surjective, there exists a map $i : P \to \mathcal{F}(S)$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "i"'] \arrow[d, "\phi"] & \\ \mathcal{F}(S) \arrow[r, two heads, "\pi"'] & \mathcal{F}(S)/Q \arrow[r] & 0 \end{tikzcd}$$

Now consider the following short exact sequence:

$$0 \longrightarrow \ker(\pi) \longrightarrow \mathcal{F}(S) \xrightarrow{\pi} \mathcal{F}(S)/Q \longrightarrow 0.$$

We know $P \cong \mathcal{F}(S)/Q$, and it is clear that $\ker(\pi) \cong Q$, so this becomes

$$0 \longrightarrow Q \longrightarrow \mathcal{F}(S) \xrightarrow{\pi} P \longrightarrow 0.$$

Thus, to show P is the direct summand of $\mathcal{F}(S)$, we need only find a section for π. However, since ϕ is an isomorphism, we have a map $i \circ \phi^{-1} : \mathcal{F}(S)/Q \to \mathcal{F}(S)$. To see that it is a section, note that

$$(\pi \circ i) \circ \phi^{-1} = \phi \circ \phi^{-1} = \mathrm{Id}$$

by the above commuting diagram. Thus our short exact sequence splits and we can conclude that $\mathcal{F}(S) \cong P \oplus Q$.

($\Longleftarrow$) Now assume P is the direct summand of a free R-module $\mathcal{F}(S)$, i.e., $\mathcal{F}(S) \cong P \oplus Q$ where Q is some R-module. Then we have natural projection and inclusion maps $\pi : \mathcal{F}(S) \to P$ and $\iota : P \hookrightarrow \mathcal{F}(S)$, respectively.

Let M, N be arbitrary R-modules with $f : P \to N$ and $v : M \to N$ R-module homomorphisms where v is surjective. Then we have

$$\begin{tikzcd} & \mathcal{F}(S) \arrow[d, "f\circ\pi"] & \\ M \arrow[r, two heads, "v"'] & N \arrow[r] & 0 \end{tikzcd}$$

Since $\mathcal{F}(S)$ is free and hence projective by the lemma above, there exists a map $g : \mathcal{F}(S) \to M$ that makes the diagram commute. That is,

$$\begin{tikzcd} & \mathcal{F}(S) \arrow[dl, dashed, "g"'] \arrow[d, "f\circ\pi"] & \\ M \arrow[r, two heads, "v"'] & N \arrow[r] & 0 \end{tikzcd}$$

commutes. But we can then simply redraw the diagram as such:

$$\begin{tikzcd} & P \arrow[dl, dashed, "g \circ \iota"'] \arrow[d, "f"] & \\ M \arrow[r, two heads, "v"'] & N \arrow[r] & 0 \end{tikzcd}$$

Thus, if this diagram commutes, P is projective. Note that the previous diagram commuting is not quite sufficient, since that diagram did not involve ι. Here, we have

$$v \circ g \circ \iota = f \circ \pi \circ \iota = f \circ Id = f,$$

where the first equality follows from the commutativity of the diagram associated to $\mathcal{F}(S)$; the second by the definitions of π and ι. Thus P is projective. $\square$

Problem 4.5.2

Let $A \in M_n(\mathbb{C})$ (with $n \geq 1$) and suppose $A^m = 0$ for some $m > 0$. Show that $A^n = 0$ and that the trace of A is zero.

Notes and Comments

Proof. **Proof 1 (just linear algebra):** Since $A^m = 0$ for some m (necessarily smaller than n), we know that A is nilpotent. Thus all the eigenvalues of A are 0 and so the trace of A, being the sum of eigenvalues of A, is also 0. Finally, $A^n = 0$ because

$$A^n = A^{n-m}A^m = A^{n-m}0 = 0.$$

Proof 2 (using some module theory): Instead, for a more interesting and satisfying answer, we could consider $\mathbb{C}^n$ as a $\mathbb{C}[x]$-module $\mathbb{C}^n_A$ with action $f \cdot v = f(A)v$. Let m be minimal such that $A^m = 0$. Then, since $A^m = 0$, the minimal polynomial of A is x^m.

As the only roots of x^m are 0 (with multiplicity), the only possible eigenvalues of A (being the roots of the minimal polynomial) are 0. That is, every eigenvalue of A is 0. Hence the characteristic polynomial of A is $\chi_A(x) = x^n$ ($x^m \mid \chi_A$ and $\chi_A \mid (x^m)^k$ for some k). As χ_A annihilates $\mathbb{C}^n_A$, $\boxed{A^n = 0}$.

The argument for trace, even in this setting, remains unchanged. However, if you prefer, consider the Jordan canonical form J of A. So $A = UJU^{-1}$ and, as the trace is the sum of the diagonal and is invariant under conjugation, $\boxed{\text{trace}(A) = \text{trace}(J) = 0}$. $\square$

Problem 4.5.3

Suppose that G is a group of order 105 and that G has a normal 3-Sylow subgroup. Show that G is cyclic.

Notes and Comments

Proof. First note that $105 = 3 \cdot 5 \cdot 7$. Let n_p denote the number of p-Sylow subgroups. By the Sylow theorems, we have $n_5 \equiv 1 \pmod 5$ and $n_5 | 21 = \frac{105}{5}$. Thus $n_5 = 1$ or 21. Similarly, $n_7 \equiv 1 \pmod 7$ and $n_7 | 15 = \frac{105}{7}$. So $n_7 = 1$ or 15.

If both $n_5 = 21$ and $n_7 = 15$, then there are $21(4) = 84$ elements of order 5 and $15(6) = 90$ elements of order 7. This is already too many. ↯ Therefore, either $n_5 = 1$ or $n_7 = 1$. Suppose $\boxed{n_5 = 1}$.

Let P be the 3-Sylow subgroup, Q a 5-Sylow subgroup, and R a 7-Sylow subgroup. Then we know that $Q \trianglelefteq G$. Then $QR \leq G$. Note that $P \cap QR = \{1\}$ since $\gcd(|P|, |QR|) = \gcd(3, 35) = 1$.

Since $P \trianglelefteq G$, we have $P(QR) \leq G$. Thus $\boxed{PQR = G}$. Then $G \cong P \rtimes_\varphi QR$, where $\varphi\colon QR \to \mathrm{Aut}(P)$ is given by $\alpha \mapsto \psi_\alpha$, where $\psi_\alpha(\beta) = \alpha\beta\alpha^{-1}$. Note that $\mathrm{Aut}(P) \cong \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. But $|QR| = 35$ and $\gcd(2, 35) = 1$, so φ is trivial. Hence $\boxed{G \cong P \times QR}$.

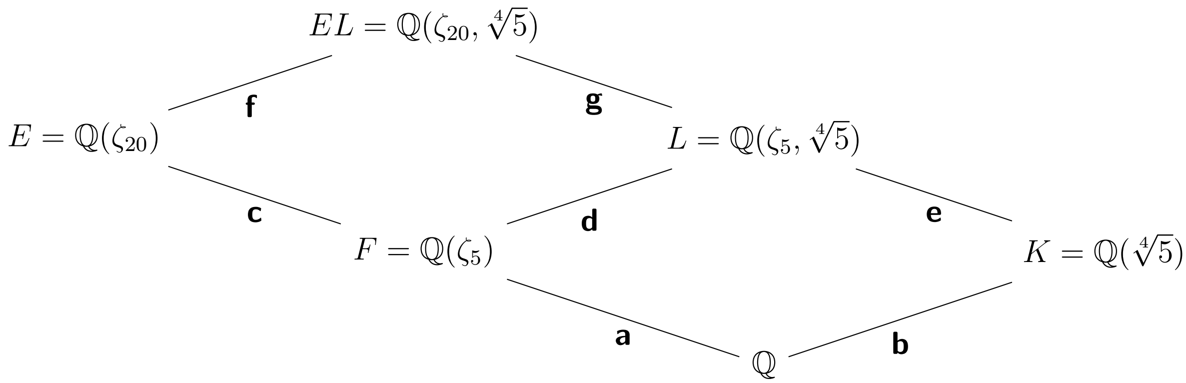
Now QR is a group of order $5 \cdot 7$, the product of 2 primes, and any such group is cyclic. Moreover $|QR| = 35$, so $QR \cong \mathbb{Z}/35\mathbb{Z}$. Thus $G \cong P \times QR \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/35\mathbb{Z} \cong \mathbb{Z}/105\mathbb{Z}$ and so G is cyclic as desired.

The same argument holds if R is normal ($n_7 = 1$) and Q is not. $\square$

Problem 4.5.4

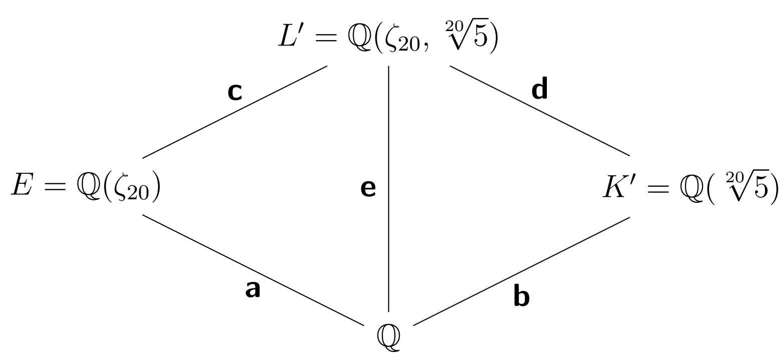
For $n \geq 1$, let ζ_n denote a primitive nth root of unity.

(a) Consider the following lattice of fields.



Determine the degrees of all extensions, giving reasons for your answers.

(b) Now consider the lattice



For each of the five extensions in this diagram, determine whether the extension is Galois or not. If the extension is Galois, determine the strongest adjective that describes the Galois group from among the following list (in ascending order):

nonabelian $\Leftarrow$ solvable $\Leftarrow$ abelian $\Leftarrow$ cyclic

and offer a brief explanation. (You may compute Galois groups to justify your assertions, but this is not required.)

Notes and Comments

Proof of (a). $\boxed{(a,b,c)}$: Observe that $[\mathbb{Q}(\sqrt[4]{5}):\mathbb{Q}] = 4$ since $x^4 - 5$ is irreducible by Eisenstein ($p = 5$). The cyclotomic extensions have degree given by $[\mathbb{Q}(\zeta_{20}):\mathbb{Q}] = \phi(20) = 8$ and $[\mathbb{Q}(\zeta_5):\mathbb{Q}] = \phi(5) = 4$. By multiplicativity of degrees in towers, this solves a, b, and c.

$\boxed{(d,e)}$: We now look to determine the degree of L/F. This requires the observation that $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}(\zeta_5)$, which is an example of the more general fact that $\mathbb{Q}(\sqrt{p^*}) \subset \mathbb{Q}(\zeta_p)$ for p prime where $p^* = \pm p$ depending on whether $p \equiv 1 \pmod 4$. With this observation, let $\alpha = \sqrt{5}$ and observe that $L = F(\sqrt{\alpha})$ so $[L:F] \leq 2$. If $[L:F] = 1$ this would imply that $F = K$, a contradiction since the former is Galois over $\mathbb{Q}$ and the latter is not. Hence $d = e = 2$.

$\boxed{(f,g)}$: To finish, we can now apply a similar argument to EL/E. Let $\alpha = \sqrt{5} \in \mathbb{Q}(\zeta_{20})$ so $EL = E(\sqrt{\alpha})$ which implies that $[EL:E] \leq 2$. Since $\mathbb{Q}(\zeta_{20})$ is abelian then every sub-extension is normal so $\mathbb{Q}(\sqrt[4]{5})$ is not a subfield. Hence $EL \neq E$ so $[EL:E] = 2$ which solves f, g

Therefore the diagram becomes

$$\begin{tikzcd}
& EL = \mathbb{Q}(\zeta_{20}, \sqrt[4]{5}) \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & & \\
E = \mathbb{Q}(\zeta_{20}) \arrow[dr, dash, "2"'] & & L = \mathbb{Q}(\zeta_5, \sqrt[4]{5}) \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
& F = \mathbb{Q}(\zeta_5) \arrow[dr, dash, "4"'] & & K = \mathbb{Q}(\sqrt[4]{5}) \arrow[dl, dash, "4"] \\
& & \mathbb{Q} &
\end{tikzcd}$$

with all degrees determined as desired. □

Proof of (b). $\boxed{a \text{ is Galois, abelian}}$: As a cyclotomic extension we have $\mathrm{Gal}(\mathbb{Q}(\zeta_{20})/\mathbb{Q}) = (\mathbb{Z}/20\mathbb{Z})^\times$. On the level of rings, we can break $\mathbb{Z}/20\mathbb{Z}$ apart as $\mathbb{Z}/20\mathbb{Z} \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ using the Chinese Remainder Theorem. Clearly the isomorphism restricts to units and hence $(\mathbb{Z}/20\mathbb{Z})^\times \cong (\mathbb{Z}/5\mathbb{Z})^\times \times (\mathbb{Z}/4\mathbb{Z})^\times$. Thus $\mathbb{Q}(\zeta_{20})/\mathbb{Q}$ is abelian but not cyclic.

$\boxed{b \text{ is not Galois}}$: This extension is not normal since it only contains the real conjugates of $\sqrt[20]{5}$.

$\boxed{c \text{ is Galois, cyclic}}$: Adjoining an n^{th} root of an element to a field with the n^{th} roots of unity is cyclic.

$\boxed{d \text{ is Galois, abelian}}$: The argument in part (a) gives that $E \cap K' = \mathbb{Q}(\sqrt{5})$. It can be checked that the subgroup $\langle (4,1),(1,3)\rangle \leq (\mathbb{Z}/5\mathbb{Z})^\times \times (\mathbb{Z}/4\mathbb{Z})^\times$ fixes $\mathbb{Q}(\sqrt{5})$. Hence by Theorem 1.12 (Lang),

$$\begin{tikzcd}
& K'F = \mathbb{Q}(\zeta_{20}, \sqrt[20]{5}) \arrow[dl, dash] \arrow[dr, dash, "\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}", "="'] & \\
E = \mathbb{Q}(\zeta_{20}) \arrow[dr, dash, "=", "\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}"'] & & K' = \mathbb{Q}(\sqrt[20]{5}) \arrow[dl, dash] \\
& E\cap K' = \mathbb{Q}(\sqrt{5}) \arrow[d, dash, "2"'] & \\
& \mathbb{Q} &
\end{tikzcd}$$

That is, the extension is abelian but not cyclic.

$\boxed{e \text{ is Galois, solvable}}$: The extension is Galois as it is the splitting field of $x^{20}-5$. Since $K' = \mathbb{Q}(\sqrt[20]{5})$ is not Galois, this extension is not abelian. However it is still solvable with a normal tower going through $\mathbb{Q}(\zeta_{20})$. $\square$

Problem 4.5.5

Finite Galois extensions.

(a) Suppose K_1/F and K_2/F are finite Galois extensions. Show that the extensions $(K_1\cap K_2)/F$ and K_1K_2/F are Galois.

(b) Prove that for any integer $n \geq 1$ there is a Galois extension $K/\mathbb{Q}$ with $[K:\mathbb{Q}] = n$.

Notes and Comments

Proof of (a). Suppose K_1, K_2 are Galois over F. Let $\alpha \in K_1 \cap K_2$ and let $f(x)$ be the minimal polynomial of α over F. Then f is separable since $\alpha \in K_1$ and K_1/F is separable. Thus every $\alpha \in K_1 \cap K_2$ is separable, so $K_1 \cap K_2$ is separable over F.

To show that $K_1 \cap K_2$ is normal over F, it suffices to show that, given α and f as above, $K_1 \cap K_2$ contains all of the roots of F. But all roots of f are in K_1 because K_1 is normal over F and, similarly, all roots of f are in K_2 since K_2 is normal. Therefore, all roots of f are in $K_1 \cap K_2$, so $K_1 \cap K_2$ is normal over F. Hence $\boxed{K_1 \cap K_2 \text{ is Galois over } F}$.

To show that K_1K_2 is Galois over F, write $K_1 = F(\alpha), K_2 = F(\beta)$ by the Primitive Element Theorem (since K_1 and K_2 are Galois extensions and hence separable). Let f_α and f_β be the minimal polynomials of α and β, respectively. Then $K_1K_2 = F(\alpha,\beta)$ is the splitting field of $f_\alpha f_\beta$, so it suffices to show that $f_\alpha f_\beta$ is separable.

Since both f_α and f_β are separable, the only way their product will not be separable is if f_α and f_β have a common root γ. Then f_α and f_β are both minimal polynomials of γ over F, so $f_\alpha = f_\beta$ and hence $K_1 = K_2$. That is, $K_1K_2 = K_1 = K_2 = F(\gamma)$ is Galois. Otherwise, f_α and f_β do not share a root so $f_\alpha f_\beta$ is separable, so K_1K_2 is separable and hence Galois over F. $\square$

Proof of (b). By Dirichlet's Theorem on Arithmetic Progressions, there is a prime p such that $p \equiv 1 \pmod{n}$ (and so that $n|(p-1)$). Let ζ_p be a primitive pth root of unity. Define $L = \mathbb{Q}(\zeta_p)$. Then $L/\mathbb{Q}$ is Galois with Galois group $(\mathbb{Z}/p\mathbb{Z})^\times$, which is cyclic of order $p-1$.

Write $p-1 = nk$. Then $k|(p-1)$ and G is abelian, so there exists a subgroup $H \leq G$ of order k. Let K be the fixed field of H. Then $[K : \mathbb{Q}] = [G : H] = (p-1)/k = n$ and hence K is the desired field. □

Problem 4.5.6

Let A be a Noetherian integral domain (which is not a field), B a commutative ring with identity.

(a) Let $a \in A$ be a nonzero, non-unit. Show that a can be written as a finite product of irreducibles.

(b) Let $\varphi : A \to B$ be a surjective ring homomorphism. Show that B is a Noetherian ring.

Notes and Comments

Proof of (a). Suppose a cannot be written this way. Then a is not irreducible, so we can write $a = a_1 b_1$ for some $a_1, b_1 \in A$, neither of which is a unit.

If both a_1 and b_1 could be written as a product of irreducibles, then $a = a_1 b_1$ also would be a product of irreducibles, a contradiction. So, without loss of generality, a_1 is not a product of irreducibles. Then a_1 is not irreducible and so $a_1 = a_2 b_2$ for some $a_2, b_2 \in A$, neither of which is a unit.

Continuing in this way, we can write $a_2 = a_3 b_3$ and, in general,

$$a_{i-1} = a_i b_i \tag{4.1}$$

with neither a_i nor b_i a unit. This yields the chain

$$(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots.$$

Now we show that this chain does not stabilize. If $(a_{i-1}) = (a_i)$ for some i, then $a_i = c a_{i-1}$. Substituting this into (4.1) yields $a_{i-1} = c a_{i-1} b_i$ where c is a unit. We are in an integral domain, so this implies $1 = c b_i$. That is, b_i is a unit. $\lightning$ Therefore, the chain

$$(a_1) \subsetneq (a_2) \subsetneq (a_3) \subsetneq \ldots$$

does not stabilize. Hence A cannot be Noetherian. □

Proof of (b). There are three equivalent definitions of a Noetherian ring and each one can be used to prove that B is Noetherian (though just one of the three is enough).

I. Let $J_1 \subseteq J_2 \subseteq \cdots$ be an ascending chain of ideals in B. Then

$$\phi^{-1}(J_1) \subseteq \phi^{-1}(J_2) \subseteq \ldots$$

is an ascending chain of ideals in A. Since A is Noetherian, this chain in A stabilizes at some m, i.e.,

$$\phi^{-1}(J_m) = \phi^{-1}(J_{m+1}) = \phi^{-1}(J_{m+2}) = \ldots.$$

Since ϕ is surjective, this implies $J_m = J_{m+1} = J_{m+2} = \ldots$, so the chain of ideals in B stabilizes.

II. Let J be an ideal in B. Then $\phi^{-1}(J)$ is an ideal in A. Since A is Noetherian, $\phi^{-1}(J)$ is finitely generated by, say, $\phi^{-1}(J) = (a_1, \dots, a_k)$. We will prove that $J = (\phi(a_1), \dots, \phi(a_k))$.

($\supseteq$) $a_i \in \phi^{-1}(J)$, so $\phi(a_i) \in J$.

($\subseteq$) Let $b \in J$. Since ϕ is surjective, $b = \phi(a)$ for some $a \in A$. Then $\phi(a) \in J$, so $a \in \phi^{-1}(J)$, so we can write

$$a = c_1 a_1 + \cdots + c_k a_k$$

for some $c_1, \dots, c_k \in A$. Then

$$b = \phi(a) = \phi(c_1)\,\phi(a_1) + \cdots + \phi(c_k)\,\phi(a_k),$$

so $b \in (\phi(a_1), \dots, \phi(a_k))$.

Therefore, J is generated by the elements $\phi(a_1), \dots, \phi(a_k)$ and we have shown that every ideal of B is finitely generated.

III. Let $\mathcal{J}$ be a non-empty collection of ideals in B. Set $\mathcal{I} = \left(\phi^{-1}(J)\right)_{J \in \mathcal{J}}$. Since A is Noetherian, $\mathcal{I}$ has a maximal element. By the Correspondence Theorem, since ϕ is surjective, $\mathcal{I}$ and $\mathcal{J}$ have identical containment relations, so $\mathcal{J}$ has a maximal element.

For any of these methods, we have that B is Noetherian. $\square$

Algebra Exam

Fall 2014

Problem 4.6.1

Let A be a 5×5 matrix over $\mathbb{C}$ with minimal polynomial $m_A(x)=x^2(x-2)^2$. What are the possible rational canonical forms and corresponding Jordan forms for A?

Notes and Comments

Proof. Note that m_A divides the characteristic polynomial χ_A and that they share the same roots. As A can be realized as a linear map, we can realize the underlying vector space as a $\mathbb{C}[x]$-module with the standard A action. Hence we have a decomposition (via the structure theorem) $f_1|f_2|\dots|f_s$ where we know that $m_A=f_s$ and $f_1f_2\dots f_s=\chi_A$.

Since A is 5×5 we must have $\deg(\chi_A)=5$. So we know that there are two possible forms for χ_A, depending on the root of the additional factor. Thus the possible characteristic polynomials are either xm_A or $(x-2)m_A$. Thus we can compute the associated canonical forms:

- **Rational Canonical Form**: The companion matrices are $C(x)=[0]$, $C((x-2))=\begin{bmatrix}2\end{bmatrix}$, and $C(x^2(x-2)^2)=C(x^4-4x^3+4x^2)=\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&1&0&-4\\0&0&1&4\end{bmatrix}$. Thus the two possible rational canonical forms are
$$R_{A_1}=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&1&0&-4\\0&0&0&1&4\end{bmatrix}\qquad\text{or}\qquad R_{A_2}=\begin{bmatrix}2&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&1&0&-4\\0&0&0&1&4\end{bmatrix}.$$
- **Jordan Forms**: We have our Jordan forms corresponding to the p-primary decompositions. From m_A, we know that we will have Jordan blocks $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ and $\begin{bmatrix}2&1\\0&2\end{bmatrix}$. The final Jordan block depends on the "extra" factor we include (either $[0]$ or $[2]$). Thus
$$J_{A_1}=\begin{bmatrix}0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&2&1\\0&0&0&0&2\end{bmatrix}\qquad\text{or}\qquad J_{A_2}=\begin{bmatrix}2&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&2&1\\0&0&0&0&2\end{bmatrix}.$$

□

Problem 4.6.2

Let R be a commutative ring with identity.

(a) Let M,N be free R-modules. Show that $M\otimes_R N$ is free.

(b) Let M, N be projective R-modules. Show that $M \otimes_R N$ is projective.

Notes and Comments

Proof of (a). A free R-module is a direct sum of copies of R. In particular, we write $M \cong \bigoplus_{i \in I} R$ and $N \cong \bigoplus_{j \in J} R$. Then, since $R \otimes_R R \cong R$,

$$M \otimes_R N \cong \left(\bigoplus_{i \in I} R \right) \otimes_R \left(\bigoplus_{j \in J} R \right) \cong \bigoplus_{(i,j) \in I \times J} R,$$

which is free. □

Proof of (b). A projective module is a direct summand of a free module. In particular, we write $M \oplus P = E$ and $N \oplus Q = F$, where E and F are free. Then, by part (a), $E \otimes_R F$ is free. Furthermore,

$$E \otimes_R F = (M \oplus P) \otimes_R (N \oplus Q) \cong (M \otimes_R N) \oplus (M \otimes_R Q) \oplus (P \otimes_R N) \oplus (P \otimes_R Q),$$

showing that $M \otimes_R N$ is a direct summand of the free module $E \otimes_R F$, whence $M \otimes_R N$ is projective. □

Problem 4.6.3

Suppose that p and q are distinct primes and that G is a group of order $p^2 q$. Show that G has either a normal p-Sylow subgroup or a normal q-Sylow subgroup.

Notes and Comments

Proof. Let n_p denote the number of p-Sylow subgroups and n_q the number of q-Sylow subgroups. Let P be a p-Sylow subgroup and Q a q-Sylow subgroup.

By the Sylow theorems, we know that $n_p \equiv 1 \pmod p$ and $n_p | [G : P]$. Since $[G : P] = \frac{|G|}{|P|} = \frac{p^2 q}{p^2} = q$, we have $n_p | q$. Thus $n_p = 1$ or $n_p = q$. Similarly, $n_q = 1, p$ or p^2.

If $n_q = 1$ then Q is normal in G and we are done. So we suppose that $n_q \neq 1$. Then $n_q = p$ or p^2 and $n_p = q$. We will show that P must be normal in this case.

Case 1: Suppose $n_q = p$. Then $p \equiv 1 \pmod q$ and so $p > q$. Thus $[G : P] = q$ is the smallest prime dividing $|G|$. Hence $P \trianglelefteq G$.

Case 2: Suppose $n_q = p^2$. Then there are $p^2(q-1)$ elements of order q and at least $p^2 - 1$ elements of order p or p^2 (namely, the nontrivial elements of P). Including the identity, we have at least $p^2 q - p^2 + p^2 - 1 + 1 = p^2 q$ elements. Thus, if there is more than one p-Sylow subgroup, there will be too many elements. Hence $n_p = 1$ and $P \trianglelefteq G$.

In any of the above cases, G has a normal p-Sylow subgroup or a normal q-Sylow subgroup. □

Problem 4.6.4

Let $\zeta_7 \in \mathbb{Z}$ be a primitive, complex 7th root of unity. Consider the lattice of fields:

$$\begin{tikzcd}[column sep=small] & L = \mathbb{Q}(\sqrt[7]{2}, \zeta_7) \arrow[dl, dash] \arrow[dr, dash] & \\ E = \mathbb{Q}(\zeta_7) \arrow[dr, dash] & & K = \mathbb{Q}(\sqrt[7]{2}) \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

(a) **Compute the degree of each extension in the diagram, justifying your answers.**

(b) **Show that L is Galois over $\mathbb{Q}$. Let $G = \mathrm{Gal}(L/\mathbb{Q})$ and H_E, H_K be the subgroups corresponding to E and K, respectively. Show explicitly that H_E and H_K are cyclic groups and compute their orders. Under the Galois correspondence, determine the subgroups $H_E \cap H_K$ and $H_E H_K$.**

(c) **Show that G is the semi-direct product of cyclic groups.**

(d) **Let $\sigma \in G$ be characterized by $\sigma(\sqrt[7]{2}) = \sqrt[7]{2}\zeta_7^5$ and $\sigma(\zeta_7) = \zeta_7^3$. What are the fixed fields corresponding to $\sigma H_E \sigma^{-1}$ and $\sigma H_K \sigma^{-1}$?**

Notes and Comments

Proof of (a). The extension $K/\mathbb{Q}$ has degree 7 as the polynomial $x^7 - 2$ is irreducible by Eisenstein ($p = 2$) and has $\sqrt[7]{2}$ as a root. The extension $E/\mathbb{Q}$ has degree $\varphi(7) = 6$ since it is a cyclotomic extension. Since 7 and 6 are relatively prime, we must have the degree of $L/\mathbb{Q}$ is 42 (and so L/E is degree 7 and L/K is 6 by multiplicativity of degrees in towers). $\square$

Proof of (b). We claim that L is the splitting field of $x^7 - 2$ over $\mathbb{Q}$. The extension is automatically separable since $\mathbb{Q}$ has characteristic 0. We noted in part (a) that $x^7 - 2$ is irreducible, so it remains to show that its roots generate L. This is clear by noting that $\zeta_7\sqrt[7]{2}$ is a root of the polynomial and $\dfrac{\zeta_7\sqrt[7]{2}}{\sqrt[7]{2}} = \zeta_7$.

To see that the fixed fields are cyclic, we note $H_K \cong (\mathbb{Z}/7\mathbb{Z})^\times$ which is cyclic (generated by 3). Similarly, H_E is cyclic because there is only one group of order 7, $\mathbb{Z}_7$.

From part (a) and basic Galois theory, we see that $E \cap K = \mathbb{Q}$ and $EK = L$. Thus, by the Galois correspondence, $H_E H_K = G$ and $H_E \cap H_K = \{e\}$. $\square$

Proof of (c). This is immediate from part (b) and the semidirect product criterion once we note that $E/\mathbb{Q}$ is normal (actually abelian) as the splitting field of $x^7 - 1$. $\square$

Proof of (d). Since H_E is normal, $\sigma H_E \sigma^{-1} = H_E$, so its fixed field remains unchanged. On the other hand, $\sigma H_K \sigma^{-1} = H_{\sigma K}$ is the fixed field of $\sigma K = \mathbb{Q}(\sqrt[7]{2}\zeta_7^5)$. $\square$

Problem 4.6.5

Finite Galois extensions.

(a) Let $K/\mathbb{Q}$ be a field extension of degree 24. Show that $x^5 + 2x^4 - 16x^3 + 6x - 10$ has no roots in K.

(b) Show that $\alpha = \sqrt{2} + \sqrt[3]{5}$ is algebraic over $\mathbb{Q}$ and determine the degree of α.

Notes and Comments

Proof of (a). By Eisenstein's Criterion ($p = 2$), $f(x) = x^5 + 2x^4 - 16x^3 + 6x - 10$ is irreducible over $\mathbb{Q}$. Adjoining any root of f results in a degree 5 extension of $\mathbb{Q}$. By multiplicativity of degrees in towers, K cannot contain a root of f since $5 \nmid 24 = [K : \mathbb{Q}]$. $\square$

Proof of (b). Note that α is algebraic over $\mathbb{Q}$ since $\mathbb{Q}(\alpha) \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$. That is, since both $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{5})$ are algebraic over $\mathbb{Q}$, their compositum $\mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$ must be as well (since algebraic extensions form a distinguished class).

Hence the degree of α over $\mathbb{Q}$ must divide $6 = [\mathbb{Q}(\sqrt{2}, \sqrt[3]{5}) : \mathbb{Q}]$. Let's play the algebra game:

$$\alpha - \sqrt{2} = \sqrt[3]{5} \Rightarrow (\alpha - \sqrt{2})^3 = 5 \Rightarrow \boxed{\alpha^3 + 6\alpha - 5 = (3\alpha^2 + 2)\sqrt{2}}.$$

This shows that $\sqrt{2} = \dfrac{\alpha^3 + 6\alpha - 5}{3\alpha^2 + 2}$. That is, $\sqrt{2} \in \mathbb{Q}(\alpha)$. Hence $\sqrt[3]{5} = \alpha - \sqrt{2} \in \mathbb{Q}(\alpha)$. Thus $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$ and so α must be of degree 6 over $\mathbb{Q}$. $\square$

Problem 4.6.6

Field extensions.

(a) Determine the number of distinct roots of the polynomial $x^n - 1$ in an algebraic closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ where p is a prime number and $n > 0$.

(b) Let $K/\mathbb{Q}$ be a finite extension of fields and let $\alpha \in K$. Suppose that there is a monic polynomial $f \in \mathbb{Z}[x]$ so that $f(\alpha) = 0$. Show that the minimal polynomial $m_{\alpha,\mathbb{Q}}(x)$ of α over $\mathbb{Q}$ lies in $\mathbb{Z}[x]$.

Notes and Comments

Proof of (a). We claim there are $\begin{cases} n \text{ roots} & \text{if } p \nmid n \\ \dfrac{n}{np^{\nu_p(n)}} \text{ roots} & \text{if } p \mid n \end{cases}$ where $\nu_p(n)$ is the largest power of p dividing n.

In the first case, we note that $x^n - 1$ and nx^{n-1} have no common roots, so $x^n - 1$ is separable. Thus we can find n distinct roots in the algebraic closure.

For the second case, let $m = \dfrac{n}{p^{\nu_p(n)}}$. Then $x^n - 1 \equiv x^m - 1 \pmod{p}$ by the Frobenius map and the result follows from the first case since m and p are relatively prime. $\square$

Proof of (b). Let S be the set of all monic polynomials in $\mathbb{Z}[x]$ for which α is a root. The hypotheses of the problem give that S is non-empty. Now let f be a minimum degree element of S.

By definition of minimal polynomial, we must have $m_{\alpha,\mathbb{Q}}|f$. In order to obtain a contradiction, assume that the degree of f is strictly greater than the degree of $m_{\alpha,\mathbb{Q}}$. Then we must have $f = m_{\alpha,\mathbb{Q}}h$ over $\mathbb{Q}[x]$, with both $m_{\alpha,\mathbb{Q}}$ and h monic non-constant polynomials (again by the definition of minimal polynomial). By Gauss' Lemma, f must also split in $\mathbb{Z}[x]$ but this contradicts the minimality of the degree of f. $\lightning$ Thus $f = m_{\alpha,\mathbb{Q}}$ since they are both monic. That is, $m_{\alpha,\mathbb{Q}} \in \mathbb{Z}[x]$. $\square$

Algebra Exam

Summer 2015

Problem 4.7.1

Let R be a commutative ring with identity (1_R) and let $S \subseteq R$ be a multiplicatively closed subset of R containing 1_R and such that $0 \notin S$. Let $S^{-1}R$ denote the ring of fractions of R with respect to S.

(a) For an R-module M, describe the construction of the module of fractions $S^{-1}M$ and its universal mapping property.

(b) Show that, if

$$L \xrightarrow{\phi} M \xrightarrow{\psi} N \tag{4.2}$$

is an exact sequence of R-modules, then the induced sequence

$$S^{-1}L \xrightarrow{S^{-1}\phi} S^{-1}M \xrightarrow{S^{-1}\psi} S^{-1}N$$

is an exact sequence of $S^{-1}R$-modules.

Notes and Comments

Proof of (a). (1) Define $S^{-1}M = \left\{\frac{m}{s} : m \in M, s \in S\right\}$, where we equate $\frac{m}{s}$ and $\frac{n}{t}$ if there is $u \in S$ such that $ut \cdot m = us \cdot n$. This is an equivalence relation.

(2) For $\frac{m}{s}, \frac{n}{t} \in S^{-1}M$, define $\frac{m}{s} + \frac{n}{t} = \frac{t \cdot m + s \cdot n}{st}$. This makes $S^{-1}M$ an abelian group.

(3) For $\frac{m}{s} \in S^{-1}M$ and $\frac{r}{t} \in S^{-1}R$, define $\frac{r}{t} \cdot \frac{m}{s} = \frac{r \cdot m}{ts}$. This makes $S^{-1}M$ an $S^{-1}R$-module.

Define the natural inclusion[18] $i : M \to S^{-1}M$ by $i(m) = \frac{m}{1}$. The universal mapping property of $S^{-1}M$ is: for any $S^{-1}R$-module N and any R-module map $f : M \to N$, there is a unique $S^{-1}R$-module map $S^{-1}f : S^{-1}M \to N$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i"'] & N \\ S^{-1}M \arrow[ur, dashed, "S^{-1}f"'] & \end{tikzcd}$$

comutes. ◻

Proof of (b). Let $\frac{l}{s} \in S^{-1}L$. Then

$$S^{-1}\psi\left(S^{-1}\phi\left(\frac{l}{s}\right)\right) = \frac{\psi(\phi(l))}{s},$$

and $\psi(\phi(l)) = 0$ as the sequence in (4.2) is exact. Thus $S^{-1}\psi \circ S^{-1}\phi = 0$, so $\boxed{\operatorname{im} S^{-1}\phi \subseteq \ker S^{-1}\psi}$.

[18]Despite being called an inclusion, it is not necessarily injective. Its kernel is a torsion submodule of M, so it is injective if and only if M is torsion-free.

Now let $\frac{m}{s} \in S^{-1}M$ with $S^{-1}\psi\left(\frac{m}{s}\right) = 0$. Then

$$\psi(m) = s \cdot \frac{\psi(m)}{s} = s \cdot S^{-1}\psi\left(\frac{m}{s}\right) = s \cdot 0 = 0.$$

Thus $m \in \ker\psi$. By the exactness of (4.2) there is $l \in L$ such that $\phi(l) = m$. Then

$$S^{-1}\phi\left(\frac{l}{s}\right) = \frac{\phi(l)}{s} = \frac{m}{s}.$$

This proves that $\frac{m}{s} \in \operatorname{im} S^{-1}\phi$, so $\boxed{\ker S^{-1}\psi \subseteq \operatorname{im} S^{-1}\phi}$. □

Problem 4.7.2

Ring theory.

(a) Let R be a PID. Show that a finitely generated R-module is projective if and only if it is free.

(b) Let R be a commutative ring with identity and $S = M_n(R)$ denote the ring of $n \times n$ matrices with entries in R. Observe that $M = R^n$ (as a column space) is a left S-module. Show that M is a projective S-module but it is not a free S-module.

Notes and Comments

Proof of (a). Every free module is projective, which takes care of one direction.[19] For the other, let P be a finitely generated R-module. Then P is a direct summand of a free R-module, say $F = P \oplus Q$ with F free. Since R is a domain, F is torsion-free. But $P \subseteq F$, so P is torsion-free. Then, since P is finitely generated and R is a PID, we conclude that P is free by the structure theorem. □

Proof of (b). Let $I \in M_n(R)$ be the identity matrix. Then $M_n(R)$ is a free module over itself with basis $\{I\}$. Let $e_i \in M = R^n$ be the column with 1 in the ith place and 0 everywhere else. As an $M_n(R)$-module, $M_n(R)$ is isomorphic to M^n (the direct sum of n copies of M) by the isomorphism

$$\Phi : I \mapsto (e_1, \dots, e_n).^{20}$$

Thus, since $M_n(R) \cong M^n$ as $M_n(R)$-modules, M is a direct summand of a free module; hence M is projective.

However, M is not a free $M_n(R)$-module. Indeed, every $b \in M = R^n$ has many non-zero matrices that annihilate it. For instance, write $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ and assume that $b_1 \neq 0$ (every matrix annihilates the zero vector). Set

$$A = \begin{pmatrix} -b_2 & b_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

[19] Take the most appropriate definition of a projective module and write a sentence to justify it. You should actually be able to do this for at least two definitions easily.

[20] We defined Φ on a basis, but we can examine what it does to any matrix. Given a matrix $A \in M_n(R)$, we have $\Phi(A) = A \cdot \Phi(I) = A \cdot (e_1, \dots, e_n) = (Ae_1, \dots, Ae_n)$, and Ae_i is just the ith column of A. Thus, Φ takes a matrix and returns the list of its columns.

Then $A \neq 0$ but $Ab = 0$. Thus, b cannot be an element of a basis and, since b was "arbitrary," M has no basis.[21] So M is not free. $\square$

Problem 4.7.3

Group theory.

(a) **Compute the order of the group** $\mathrm{GL}_n(\mathbb{F}_p)$**.**

(b) **Show there is a natural short exact sequence of groups**

$$1 \longrightarrow \mathrm{SL}_n(\mathbb{F}_p) \longrightarrow \mathrm{GL}_n(\mathbb{F}_p) \longrightarrow \mathbb{F}_p^\times \longrightarrow 1$$

and that the sequence splits.

(c) **Given that the sequence splits, we know that** $\mathrm{GL}_n(\mathbb{F}_p)$ **is a semidirect product of** $\mathrm{SL}_n(\mathbb{F}_p)$ **and** $\mathbb{F}_p^\times$**. Describe this isomorphism.**

(d) **Note that part (b) implies that** $[\mathrm{GL}_n(\mathbb{F}_p) : \mathrm{SL}_n(\mathbb{F}_p)] = p - 1$**, so that the size of the Sylow** p**-subgroups is the same in** $\mathrm{SL}_n(\mathbb{F}_p)$ **and** $\mathrm{GL}_n(\mathbb{F}_p)$**. Show that every Sylow** p**-subgroup of** $\mathrm{GL}_n(\mathbb{F}_p)$ **is actually a subgroup of** $\mathrm{SL}_n(\mathbb{F}_p)$**.**

Notes and Comments

Proof of (a). An element of $\mathrm{GL}_n(\mathbb{F}_p)$ can be identified with its matrix representation via the standard basis for $\mathbb{F}_p^n$. To be in the general linear group means that this matrix is non-singular. Hence the columns must be linearly independent.

To count the elements of $\mathrm{GL}_n(\mathbb{F}_p)$, we count the number of linearly independent sets of n vectors in $\mathbb{F}_p^n$. After $j-1$ linearly independent column vectors, $\{v_1, ..., v_{j-1}\}$, have been chosen, we must choose v_j from $\mathbb{F}_p^n \setminus \mathrm{span}_{\mathbb{F}_p}\{v_1, ..., v_{j-1}\}$. Hence, there are $p^n - p^{j-1} = p^{j-1}(p^{n-j+1} - 1)$ choices for the j^{th} column. Multiplying these numbers together and playing around with indicies, we get a clean-ish expression for the order of $\mathrm{GL}_n(\mathbb{F}_p)$:

$$\begin{aligned}
|\mathrm{GL}_n(\mathbb{F}_p)| &= \prod_{j=1}^{n} p^{j-1}(p^{n-j+1} - 1) = \left(\prod_{j=1}^{n} p^{j-1}\right)\left(\prod_{j=1}^{n}(p^{n-j+1} - 1)\right) \\
&= p^{\sum_{j=1}^{n-1} j}\left(\prod_{(n-j+1)=1}^{n}(p^{(n-j+1)} - 1)\right) = p^{\frac{n(n-1)}{2}}\left(\prod_{j=1}^{n}(p^j - 1)\right).
\end{aligned}$$

Thus $\boxed{|\mathrm{GL}_n(\mathbb{F}_p)| = p^{\frac{n(n-1)}{2}}\left(\prod_{j=1}^{n-1}(p^j - 1)\right)}$. $\square$

[21] The assumption on b_1 is rather stricter than we require but it's easier to write down. Indeed, the correct assumption is that *some* $b_i \neq 0$. Then we move to the ith row of A and do a similar trick.

Proof of (b). By definition, $\mathrm{SL}_n(\mathbb{F}_p)$ is the subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ whose elements have determinant equal to one. That is, $\mathrm{SL}_n(\mathbb{F}_p)$ is the kernel of the surjective group homomorphism $\det : \mathrm{GL}_n(\mathbb{F}_p) \to \mathbb{F}_p^\times$. Equivalently, the following sequence is exact:

$$1 \longrightarrow \mathrm{SL}_n(\mathbb{F}_p) \xrightarrow{\iota} \mathrm{GL}_n(\mathbb{F}_p) \xrightarrow{\det} \mathbb{F}_p^\times \longrightarrow 1$$

where $\iota : \mathrm{SL}_n(\mathbb{F}_p) \to \mathrm{GL}_n(\mathbb{F}_p)$ is the inclusion map.

A section of det is given by the map $\cdot \oplus I : \mathbb{F}_p^\times \to \mathrm{GL}_n(\mathbb{F}_p)$. Given $u \in \mathbb{F}_p^\times$,

$$u \mapsto u \oplus I := \begin{pmatrix} u & 0 \\ 0 & I_{n-1} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{F}_p).$$

Then $\cdot \oplus I$ is indeed a group homomorphism because we're multiplying diagonal matrices. As $\det(u \oplus I) = u \cdot 1^{n-1} = u$, $\cdot \oplus I$ is indeed a section of det. Thus the short exact sequence splits as desired. □

Proof of (c). Since the short exact sequence splits via a section, we know that there is an action $\varphi : \mathbb{F}_p^\times \to \mathrm{Aut}(\mathrm{SL}_n(\mathbb{F}_p))$ such that $\mathrm{GL}_n(\mathbb{F}_p) \cong \mathrm{SL}_n(\mathbb{F}_p) \rtimes_\varphi \mathrm{Aut}(\mathbb{F}_p^\times)$. Specifically, for $u \in \mathbb{F}_p^\times$, the action is given by

$$\varphi_u(A) = (u \oplus I)A(u \oplus I)^{-1} = (u \oplus I)A(u^{-1} \oplus I).$$[22]

So the multiplication on $\mathrm{SL}_n(\mathbb{F}_p) \rtimes_\varphi \mathrm{Aut}(\mathbb{F}_p^\times)$ is

$$(A, u) \cdot (B, v) = (A\varphi_u(B), uv) = \left(A(u \oplus I)B(u^{-1} \oplus I), uv\right).$$

Consider the map $\mu : \mathrm{SL}_n(\mathbb{F}_p) \rtimes \mathbb{F}_p^\times \to \mathrm{GL}_n(\mathbb{F}_p)$ given by $\boxed{\mu(A, u) = A(u \oplus I)}$. It follows that

$$\begin{aligned} \mu((A, u) \cdot (B, v)) &= \mu\left(A(u \oplus I)B(u^{-1} \oplus I), uv\right) \\ &= A(u \oplus I)B(u^{-1} \oplus I)(uv \oplus I) \\ &= A(u \oplus I)B(u^{-1}uv \oplus I) \\ &= \mu(A, u)\mu(B, v). \end{aligned}$$

Therefore μ is a homomorphism.

Now suppose $(A, u) \in \ker \mu$. Then $\mu(A, u) = I_n$. Applying det, we have

$$1 = \det(I_n) = \det(\mu(A, u)) = \det A \det(u \oplus I) = u \det A = u$$

since $A \in \mathrm{SL}_n(\mathbb{F}_p)$. Thus $u = 1$ and so $I_n = \mu(A, u) = A$. Hence μ is injective.

To show that μ is surjective, let $B \in \mathrm{GL}_n(\mathbb{F}_p)$. Define $u := \det B$ and write $B = (b_1 \; b_2 \; \cdots \; b_n)$ where b_i is the ith column of B. Then

$$\mu\left((u^{-1}b_1 \; b_2 \; \cdots \; b_n), u\right) = (u^{-1}b_1 \; b_2 \; \cdots \; b_n)(u \oplus I) = B.$$

Thus μ is surjective and hence the desired isomorphism. □

Proof of (d). By the Sylow theorems, there exists a Sylow p-subgroup of $\mathrm{SL}_n(\mathbb{F}_p)$ which we will call P. Hence P is also a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$.

Again using the Sylow theorems, we know that the set of Sylow p-subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$ form a transitive G-set under conjugation. Hence, given any other Sylow p-subgroup P' of $\mathrm{GL}_n(\mathbb{F}_p)$, there exists $A \in \mathrm{GL}_n(\mathbb{F}_p)$ such that $\boxed{P' = APA^{-1}}$.

Let $B' \in P'$. Then there is some $B \in P$ such that $B' = ABA^{-1}$. Because the determinant is multiplicative, $\det(B') = \det(ABA^{-1}) = \det(B) = 1$. Hence $P' \leq \mathrm{SL}_n(\mathbb{F}_p)$ as desired. □

[22]Since the first map in the short exact sequence is an inclusion.

Problem 4.7.4

Let R be a (commutative) domain with 1 containing $\mathbb{C}$ as a subring. Suppose R is a finite-dimensional $\mathbb{C}$-vector space. Show that $R = \mathbb{C}$.

Notes and Comments

Proof. Consider the map $m_\alpha : R \to R$ given by $\beta \mapsto \alpha\beta$ for $\alpha \neq 0$. Since R is a vector space, m_α is a linear map. As R is a domain, m_α must be an isomorphism. Thus α is invertible.

Since every nonzero element of R is a unit, R must be a field. By finite dimensionality as a vector space, R is now a finite field extension of $\mathbb{C}$. However, $\mathbb{C}$ is algebraically closed, and so $R = \mathbb{C}$. □

Problem 4.7.5

Let K_1 and K_2 be finite fields of characteristic p. Let $q_1 = \#K_1$ and $q_2 = \#K_2$. Recall that ring homomorphisms are required to map 1 to 1. Show that the following are equivalent:

(i) **There is a ring homomorphism $K_1 \to K_2$;**

(ii) **There is an injective group homomorphism $K_1^\times \to K_2^\times$; and**

(iii) **q_2 is a power of q_1.**

Notes and Comments

Proof. **(i) $\Rightarrow$ (ii):** Assume there is a ring homomorphism $\phi\colon K_1 \to K_2$. Because K_1 is a field, ϕ is injective. Since ϕ is injective, ϕ maps $K_1^\times$ into $K_2^\times$. Then, since ϕ preserves multiplication, it is a group homomorphism.

(ii) $\Rightarrow$ (iii): Assume there is an injective group homomorphism $K_1^\times \to K_2^\times$. Then $\#K_1^\times$ divides $\#K_2^\times$. That is, $q_1 - 1$ divides $q_2 - 1$.

Let $q_1 = p^a$ and $q_2 = p^b$. We will show that $\gcd(a, b) = a$ using the following cute lemma:

Lemma 4.7.1 *For any non-negative integers n, y, z, we have* $\gcd(n^y - 1, n^z - 1) = n^{\gcd(y,z)} - 1$.

Proof of Cute Lemma. Let $X = \gcd(n^y - 1, n^z - 1)$ and $x = \gcd(y, z)$. Observe that $n^x - 1$ divides both $n^y - 1$ and $n^z - 1$, so $\boxed{n^x - 1 \mid X}$.

To see the other divisibility relation, note that n and X are relatively prime and select the smallest e so that $n^e \equiv 1 \pmod{X}$. Then $e \mid y$ and $e \mid z$, so $e \mid x$ as well. Thus $n^e - 1 \mid n^x - 1$ and hence $\boxed{X \mid n^x - 1}$. Thus $\boxed{X = n^x - 1}$. □

Applying the lemma (with $n = p$, $y = a$, and $z = b$), we see that $\gcd(a, b) = a$ (since $q_1 - 1 \mid q_2 - 1$). Hence $a \mid b$ and so $q_2 = p^b$ is a power of $q_1 = p^a$.

(iii) $\Rightarrow$ (i): Assume q_2 is a power of q_1, say $q_2 = q_1^d$. Write $q_1 = p^{e_1}$ and $q_2 = p^{e_2}$, so $e_2 = de_1$ and $q_2 = p^{de_1} = q_1^d$. We claim that $x^{q_1} - x$ divides $x^{q_2} - x$.

Let α be a root of $x^{q_1} - x$ (in some splitting field). Then $\alpha^{q_1} = \alpha$, so $\alpha^{q_2} = \alpha^{q_1^d} = (((\alpha^{q_1})^{q_1})^{\cdots})^{q_1} = \alpha$. Therefore α is a root of $x^{q_2} - x$. Since every root of $x^{q_1} - x$ is also a root of $x^{q_2} - x$, it follows that $x^{q_1} - x$ divides $x^{q_2} - x$.

Thus, since $x^{q_2} - x$ splits in K_2, $x^{q_1} - x$ also splits in K_2. But K_1 is a splitting field of $x^{q_1} - x$, so K_1 is isomorphic to a subfield of K_2 (namely, the smallest subfield of K_2 in which $x^{q_1} - x$ splits). □

Problem 4.7.6

The polynomial $f(X) = X^6 - 4X^3 + 1$ is irreducible over $\mathbb{Q}$. Let K be a splitting field for f. Show that $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3 \times \mathbb{Z}/2\mathbb{Z}$.

Notes and Comments

Proof. First write

$$f(X) = X^6 - 4X^3 + 1 = (X^3)^2 - 4(X^3) + 1 = (X^3 - 2 - \sqrt{3})(X^3 - 2 + \sqrt{3})$$

in order to find the roots of f. Let $\alpha = \sqrt[3]{2+\sqrt{3}}$ and $\beta = \sqrt[3]{2-\sqrt{3}}$. Furthermore, let $\omega = \frac{1+\sqrt{3}i}{2}$ be a primitive cube root of unity. Then the roots of f in K are $\{\alpha, \alpha\omega, \alpha\omega^2, \beta, \beta\omega, \beta\omega^2\}$. Label these as $1, 2, 3, 4, 5, 6$, respectively, to obtain an embedding of G in S_6.

Notice that $\alpha\beta = \sqrt[3]{(2+\sqrt{3})(2-\sqrt{3})} = 1$. So, for any $g \in G = \mathrm{Gal}(K/\mathbb{Q})$, we have $g(\alpha)g(\beta) = g(\alpha\beta) = g(1) = 1$. Hence $\boxed{g(\beta) = 1/g(\alpha)}$. Thus g is completely determined by $g(\alpha)$ and $g(\omega)$.

Step 1: Find the generators of G that fix ω.

- Define $\sigma \in G$ by $\sigma(\alpha) = \alpha\omega$ and $\sigma(\omega) = \omega$. Then σ corresponds to the permutation $(123)(465)$.
- Define $\tau \in G$ by $\tau(\alpha) = \beta$ and $\tau(\omega) = \omega$. So $\tau(\beta) = \tau(1/\alpha) = 1/\tau(\alpha) = 1/\beta = \alpha$. Thus τ corresponds to the permutation $(14)(25)(36)$.

We claim that σ and τ generate all elements of G that fix ω. To show this, we must show that any element $g \in G$, which sends α to a root of f such that $g(\omega) = \omega$, is generated by σ and τ. There are 6 cases to check:

(1) If $g(\alpha) = \alpha$, then $g = \mathrm{id}$.

(2) If $g(\alpha) = \alpha\omega$, then $g = \sigma$.

(3) If $g(\alpha) = \alpha\omega^2$, then $g = \sigma^2$.

(4) If $g(\alpha) = \beta$, then $g = \tau$.

(5) If $g(\alpha) = \beta\omega$, then $g = \tau \circ \sigma = \sigma^2 \circ \tau$.

(6) If $g(\alpha) = \beta\omega^2$, then $g = \tau \circ \sigma^2 = \sigma \circ \tau$.

Thus the claim is true. Hence the subgroup $H = \langle \sigma, \tau \rangle$ of G generated by σ and τ has 6 elements (namely those listed above) since $\sigma^3 = \tau^2 = \mathrm{id}$. Since $\sigma \circ \tau \neq \tau \circ \sigma$, H is not abelian. Hence $\boxed{H \cong S_3}$.

Step 2: Find the generator of G that fixes α.

Let $\gamma \in G$ such that $\gamma(\alpha) = \alpha$ and $\gamma(\omega) = \omega^2$.[23] Then γ corresponds to the permutation $(23)(56)$. This is the only element that fixes α but not ω, so any element in G is a composition of σ, τ and γ, i.e. $G = \langle \sigma, \tau, \gamma \rangle$.

Step 3: Find $\delta \in G$ such that $G = \langle \sigma, \tau, \delta \rangle$ where δ commutes with σ and τ and has order 2. (As a consequence of doing so, we would have $G \cong H \times \langle \delta \rangle \cong S_3 \times \mathbb{Z}/2\mathbb{Z}$ via $h \circ \delta \mapsto (h, \delta)$.)

[23] γ is complex conjugation.

We claim $\delta = (\tau \circ \gamma)$ satisfies the above conditions. By permutation composition, δ corresponds to the permutation $(14)(26)(35)$ (which clearly has order 2). Moreover,

$$G = \langle \sigma, \tau, \gamma \rangle = \langle \sigma, \tau, \tau \circ \gamma \rangle = \langle \sigma, \tau, \delta \rangle.$$

Therefore, it suffices to show that $\delta = (14)(26)(35)$ commutes with $\sigma = (123)(465)$ and $\tau = (14)(25)(36)$.

We can do this by direct computation:

- $\delta \circ \sigma = \sigma \circ \delta$ since

$$(14)(26)(35) \circ (123)(465) = (163425) = (123)(465) \circ (14)(26)(35).$$

- $\delta \circ \tau = \tau \circ \delta$ since

$$(14)(26)(35) \circ (14)(25)(36) = (23)(56) = (14)(25)(36) \circ (14)(26)(35).$$

Thus we have $\boxed{G \cong S_3 \times \mathbb{Z}/2\mathbb{Z}}$ as desired. $\square$

Algebra Exam

Fall 2015

Problem 4.8.1

Let $M = \mathbb{Z}^n$ and denote by $pM = (p\mathbb{Z})^n$. Suppose that L is a submodule of M with $pM \subset L \subseteq M$.

(a) Show that L is a free $\mathbb{Z}$-module of rank n.

(b) Show that index $[M : L]$ is finite, and in terms of the index describe the invariant factors (or elementary divisors) of L in M as a $\mathbb{Z}$-module.

(c) Now we fix $n = 2$: Let $pM = (p\mathbb{Z})^2 \subsetneq L \subsetneq \mathbb{Z}^2$. Count the number of possible submodules L.

Notes and Comments

Proof of (a). Since L is a submodule of a free $\mathbb{Z}$-module of rank n (namely, M) and $\mathbb{Z}$ is a PID, L must also be free of rank $m \leq n$. Now pM is also a free $\mathbb{Z}$-module, so it is free and its rank is n.[24] Thus pM is a submodule of a free $\mathbb{Z}$-module (L), so pM is free of rank $n \leq m$. Hence $m = n$ and so L has rank n. □

Proof of (b). Note that $[M : pM] = p^n$ since $\mathbb{Z}^n/(p\mathbb{Z})^n \cong (\mathbb{Z}/p\mathbb{Z})^n$ (which has size p^n). Now

$$[M : L][L : pM] = [M : pM] = p^n$$

and thus $[M : L] \mid p^n < \infty$. So we have $[M : L] = p^m$ with $m < n$ is finite.

The Invariant Factor Theorem says that there exists a basis $\{b_1, \dots, b_n\}$ for M such that $\{a_1b_1, \dots, a_nb_n\}$ is a basis for L where $a_1 \mid a_2 \mid \cdots \mid a_n$ are the invariant factors. Thus

$$M/L \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_n\mathbb{Z},$$

and so $[M : L] = a_1a_2 \cdots a_n = p^m$. Note that $\{pb_1, \dots, pb_n\}$ is a basis for pM, so we must have $a_i | p$ for all i. Thus we have $a_1 = \cdots = a_m = 1$ and $a_{m+1} = \cdots = a_n = p$. □

Proof of (c). Since p is prime, $[M : L] \mid p^2$, so $[M : L] = 1, p$ or p^2. Note that $[M : L] \neq 1, p^2$ since $L \neq M, pM$. Indeed, if $[M : L] = 1$ then $L = M$; if $[M : L] = p^2 = [M : pM]$ then $[L : pM] = 1$ and so $L = pM$. $\lightning$ Thus $\boxed{[M : L] = p}$. The only possible invariant factors are 1 and p, so we have $a_1 = 1$ and $a_2 = p$.

Let L be such that $(p\mathbb{Z})^2 \subsetneq L \subsetneq \mathbb{Z}^2$. Consider a $\mathbb{Z}$-basis $\{b_1, b_2\}$ of $\mathbb{Z}^2$ such that $\{b_1, pb_2\}$ is a basis for L. Then the change of basis matrix from $\{b_1, b_2\}$ to $\{e_1, e_2\}$ is

$$Q_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

That is, $b_1 = ae_1 + be_2$ and $b_2 = ce_1 + de_2$. Moreover, since these bases generate the same lattice, Q_1 must have $\det Q_1 = \pm 1$. Hence $\boxed{\gcd(a, b) = \gcd(c, d) = 1}$ $(*)$ (otherwise, some prime $q \mid ad - bc = \pm 1$).

We now have that

$$L = \operatorname{span}\{b_1, pb_2\} = \operatorname{span}\{ae_1 + ce_2, pbe_1 + pde_2\} = \operatorname{span}\{Q_2e_1, Q_2e_2\},$$

[24] A basis is given by $pe_1, \dots, pe_n$ where $\{e_i\}$ is the standard basis for $\mathbb{Z}^n$.

where $Q_2 = \begin{pmatrix} a & pb \\ c & pd \end{pmatrix}$. Note that $\det Q_2 = p \det Q_1 = \pm p$. Thus, by $(*)$, either $\gcd(a, pb) = 1$ or $\gcd(c, pd) = 1$.

- Assume $\gcd(a, pb) = 1$. By the Division Algorithm, $a = q(pb) + 1$. So, by elementary integer column operations[25], we obtain
$$\begin{pmatrix} a & pb \\ c & pd \end{pmatrix} \to \begin{pmatrix} 1 & pb \\ c - q(pd) & pd \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ t & \pm p \end{pmatrix}$$
where $t = c - qpd$. We know that $pd - pb(c - qpd) = \pm p$ since the only operations used involved integer multiples of columns being added to each other and hence the determinant is preserved. By further application of such operations, we may assume $\boxed{0 \leq t < p}$ and, changing the determinant, obtain the form
$$Q_3 = \begin{pmatrix} 1 & 0 \\ t & p \end{pmatrix}.$$
Since only column operations were used, we have
$$L = \operatorname{span}\{Q_2e_1, Q_2e_2\} = \operatorname{span}\{Q_3e_1, Q_3e_2\} = \operatorname{span}\{e_1 + te_2, pe_2\}$$
for $0 \leq t < p$.

- Assume $\gcd(c, pd) = 1$. Analogously, we obtain
$$Q_4 = \begin{pmatrix} p & t \\ 0 & 1 \end{pmatrix}$$
from elementary integer column operations on Q_2 where $0 \leq t < p$. Thus
$$L = \operatorname{span}\{Q_4e_1, Q_4e_2\} = \operatorname{span}\{pe_1, te_1 + e_2\}.$$

In each case above, we generate p distinct submodules of $\mathbb{Z}^2$ containing $(p\mathbb{Z})^2$. Hence there are $\boxed{2p}$ different submodules such that $(p\mathbb{Z})^2 \subsetneq L \subsetneq \mathbb{Z}^2$. □

Problem 4.8.2

Let F be a field and $f \in F[x]$ a polynomial with $\deg(f) = n \geq 1$, and suppose that $f = f_1^{e_1} \cdots f_t^{e_t}$ is the (unique) factorization of f into a product of irreducibles (i.e., f_i is irreducible and $\gcd(f_i, f_j) = 1$ for $i \neq j$). Let
$$S = \{A \in M_n(F) \mid \chi_A = f\},$$
where χ_A denotes the characteristic polynomial of A.

(a) Show that S is the union of a finite number of similarity (conjugacy) classes.

(b) Show that the number of similarity classes in S is one if and only if $e_1 = e_2 = \cdots = e_t = 1$.

Notes and Comments

[25] So we may swap columns, multiply a column by -1, or add an integer multiple of one column to another.

Proof of (a). We know that S consists of similarity classes as similar matrices have the same characteristic polynomial. To show that S contains only finitely many similarity classes, note that each matrix $M \in S$ is in the same similarity class as a matrix in rational canonical form. Thus it suffices to show there are only finitely many possibilities for the set of invariant factors, since these determine the rational canonical form uniquely (up to reordering).

Any given set of invariant factors are of the form $g_1, \ldots, g_k$, where $g_1 \mid g_2 \mid \cdots \mid g_k$ and $g_1 \cdots g_k = f$. Since f has only finitely many factors and $k \leq \sum_{i=1}^{t} e_i$, there are only a finite set of invariant factors. Thus there are a finite number of matrices in rational canonical form with characteristic polynomial f. Hence S is a union of finitely many similarity classes. □

Proof of (b). ($\Rightarrow$) Assume that the number of similarity classes in S is one. To obtain a contradiction, suppose that $e_i > 1$ for some i. Consider the matrices A_1, A_2 where A_1 has invariant factors $g_1 = f_i, g_2 = f/f_i$ and A_2 is the companion matrix of f, i.e. it corresponds to the single invariant factor f. Then, since the rational canonical form of a matrix is uniquely determined by the list of invariant factors, A_1 and A_2 have different rational canonical forms and therefore are not in the same similarity class. $\lightning$

($\Leftarrow$) Assume $e_1 = \cdots = e_n = 1$. Suppose there are invariant factors g_1 and g_2 for some similarity class in S. Then we may assume $g_1 \mid g_2$ and $g_1 g_2 \mid f$. Since $f = f_1 \cdots f_n$, we have $f_i \mid g_1$ for some i. Thus $f_i \mid g_2$ as well. However, this means $f_i^2 \mid f$ but $e_i = 1$ and the factorization of f is unique. $\lightning$ Thus all matrices in S have the same rational canonical form.[26] Therefore S contains only one similarity class. □

Problem 4.8.3

Simple groups.

(a) Let G be a finite group and let H be a subgroup with index $[G : H] = n \geq 2$. Show that if G is a simple group, then there exists an injective group homomorphism $\varphi : G \to S_n$ where S_n is the symmetric group on n letters.

(b) Show that no group of order $96 = 3 \cdot 2^5$ is simple.

Notes and Comments

Proof of (a). Write $G/H = \{g_1 H = H, g_2 H, \ldots, g_n H\}$. Define $\varphi : G \to S_n$ by $\varphi(g) = \sigma_g$ where σ_g is the permutation of G/H given by $\sigma_g(g_i H) = g g_i H$. Then

$$\varphi(g_1 g_2) = \sigma_{g_1 g_2} = \sigma_{g_1} \circ \sigma_{g_2} = \varphi(g_1) \circ \varphi(g_2)$$

and so φ is a group homomorphism.

We now show that φ is injective. Indeed, since G is simple, it has no nontrivial proper normal subgroups. As $\ker \varphi \lhd G$, $\ker \varphi$ is either trivial or all of G. However, the latter cannot hold since

$$g \in \ker \varphi \Rightarrow \sigma_g = \mathrm{Id} \Rightarrow gH = H \Rightarrow g \in H$$

and $H < G$ is a proper subgroup. Thus $\boxed{\ker \varphi = \{e\}}$ and so φ is an injective group map as desired. □

Proof of (b). Suppose there was a simple group G of order 96. Then G has a Sylow-2 subgroup S of order $32 = 2^5$. Thus $[G : S] = 3$. By part (a), G embeds into S_3, a group of order 6. $\lightning$ Thus no such simple group exists. □

[26] Namely the companion matrix $C_{(f)}$ of f.

Problem 4.8.4

Let G be a finite group. Let $G' \leq G$ be the subgroup generated by all elements of the form $\sigma\tau\sigma^{-1}\tau^{-1}$ for $\sigma, \tau \in G$. (We call G' the *commutator* subgroup of G.)

(a) Show that $G' \trianglelefteq G$ is normal and that G' is the smallest normal subgroup of G such that G/G' is abelian.

(b) Now suppose $K \supseteq F$ is a Galois extension with Galois group G. Show that the fixed field of G' corresponds to the maximal subextension $K \supseteq M \supseteq F$ such that M/F is Galois with abelian Galois group.

Notes and Comments

Proof of (a). To show G' is normal in G, we must show that, for all $g \in G, \sigma \in G'$, we have $g\sigma g^{-1} \in G'$. We know that $a := g\sigma g^{-1}\sigma^{-1} \in G'$ by definition. Then $g\sigma g^{-1} = a\sigma \in G'$ since $a, \sigma \in G'$. Thus G' is normal.

Now we show that G/G' is abliean. Let $g_1, g_2 \in G$. Then, since $g_2^{-1}g_1^{-1}g_2g_1 \in G'$,

$$g_1G' \cdot g_2G' = g_1g_2G' = g_1g_2(g_2^{-1}g_1^{-1}g_2g_1)G' = g_2g_1G' = g_2G' \cdot g_1G'.$$

Finally, to show that G' is the smallest such subgroup, let $N \trianglelefteq G$ be another normal subgroup of G with G/N abelian. We will show that $G' \leq N$.

Let $\sigma = aba^{-1}b^{-1} \in G'$. Then, since G/N is abelian, we have $a^{-1}b^{-1}N = b^{-1}a^{-1}N$, so $aba^{-1}b^{-1}N = \sigma N = N$. Thus $\sigma \in N$ and so all generators of G' are in N, i.e., $G' \leq N$. Therefore G' is the smallest normal subgroup of G such that G/G' is abelian. □

Proof of (b). Let M be the fixed field of G'. Since $M \subseteq K$ and K/F is separable, we know that M/F is separable. Also, M/F is normal because $G' \trianglelefteq G$. Thus M/F is Galois and $\mathrm{Gal}(M/F) \cong \mathrm{Gal}(K/F)/\mathrm{Gal}(K/M) \cong G/G'$, which is abelian from part (a).

Now we show that M/F is the maximal such Galois extension. Let $M' \supseteq M$ be another such extension, corresponding to a subgroup $H \leq G$. By the Galois correspondence, $H \leq G'$. Since M'/F is Galois, $H \trianglelefteq G$. Moreover, $\mathrm{Gal}(M'/F) \cong G/H$ is abelian by hypothesis. By part (a), we also have $G' \leq H$ and so $G' = H$. Thus $M = M'$ and hence M is the maximal subextension such that M/F is Galois with abelian Galois group. □

Problem 4.8.5

Let $f(X) = (X^4 - 2X^2 - 1)(X^2 - 2)(X^2 + 1)$.

(a) Show that $g(X) = X^4 - 2X^2 - 1$ is irreducible over $\mathbb{Q}$.

(b) Exhibit a splitting field $K = K_f$ for f and show that it is equal to the splitting field for g.

(c) Show that $\mathrm{Gal}(f)$ is nonabelian.

Notes and Comments

Proof of (a). By the Rational Root Test, the only roots of can be ± 1. As neither of these are roots, if g has a nontrivial factorization over $\mathbb{Q}$, it must have two quadratic factors.

Suppose $(X^2 + a_1 X + a_0)(X^2 + b_1 X + b_0) = g(X)$. Then

$$g(X) = X^4 + (a_1 + b_1)X^3 + (a_0 + b_0 + a_1 b_1)X^2 + (a_1 b_0 + a_0 b_1)X + a_0 b_0$$

where $\begin{cases} a_1 + b_1 = 0 \\ a_0 + b_0 + a_1 b_1 = -2 \\ a_1 b_0 + a_0 b_1 = 0 \\ a_0 b_0 = -1 \end{cases}$.

Combining the first and third equations, we get $\boxed{a_1 b_0 - a_0 a_1 = 0}$. Then either $a_1 = 0$ or $a_0 = b_0$.

- If $a_1 = 0$ then $a_0 + b_0 = -2$ by the second equation. Substituting into the fourth equation, we have

  $$a_0 b_0 = a_0(-a_0 - 2) = -a_0^2 - 2a_0 = -1,$$

  and so $a_0^2 + 2a_0 - 1 = 0$. However, there is no $a_0 \in \mathbb{Q}$ that satisfies this equation since $Y^2 + 2Y - 1$ is irreducible over $\mathbb{Q}$ by the Rational Root Test. $\lightning$

- If $a_0 = b_0$. Then we have $a_0^2 = a_0 b_0 = -1$. However, $Y^2 + 1$ is irreducible over $\mathbb{Q}$. $\lightning$

Therefore g cannot be factored into quadratics. Hence g is irreducible over $\mathbb{Q}$. $\square$

Proof of (b). The roots of g are $\pm\alpha$ and $\pm\beta$, where $\alpha = \sqrt{1+\sqrt{2}}$ and $\beta = \sqrt{1-\sqrt{2}}$. Thus the splitting field for g is $K_g = \mathbb{Q}(\alpha, \beta)$.

The roots of f are $\pm\alpha, \pm\beta, \sqrt{2}$, and i. Thus $K_f = K_g(\sqrt{2}, i)$. To show $K_f = K_g$, we must show that $\sqrt{2}, i \in K_g$. Indeed, note that $\sqrt{2} = \alpha^2 - 1 \in K_g$ and $i = \alpha\beta \in K_g$ since $\alpha, \beta \in K_g$. Thus $K_f = K_g$. $\square$

Proof of (c). Consider $\sigma, \tau \in \mathrm{Gal}(f)$ where:

- $\sigma(\alpha) = \beta$ and $\sigma(\beta) = \alpha$,
- $\tau(\alpha) = -\alpha$ and $\tau(\beta) = \beta$.

Then

$$\sigma \circ \tau(\alpha) = \sigma(-\alpha) = -\beta \text{ and } \tau \circ \sigma(\alpha) = \tau(\beta) = \beta.$$

Thus $\sigma\tau \neq \tau\sigma$ and so $\mathrm{Gal}(f)$ is nonabelian. $\square$

Problem 4.8.6

Let F be a field with $\mathrm{char}\, F \neq 2$. Let $a \in F^\times \backslash F^{\times 2}$, and let $K = F(\sqrt{a})$. Then $\mathrm{Gal}(K/F) = \langle\sigma\rangle \simeq \mathbb{Z}/2\mathbb{Z}$.

Let $b = x + y\sqrt{a} \in K^\times \setminus K^{\times 2}$ with $x, y \in F$ and let $L = K(\sqrt{b})$. Show that L is Galois over F if and only if $b\sigma(b) = x^2 - ay^2 \in K^{\times 2}$.

Notes and Comments

Proof. ($\Rightarrow$) Assume L/F is Galois. Then, L/F is a normal extension and so L contains all roots of f, where f is the minimal polynomial of $\sqrt{b}$ over F. In particular, $\sqrt{\sigma(b)} \in L$ and so $\sqrt{b\sigma(b)} = \sqrt{x^2 - ay^2} \in L$.

Now $\sqrt{x^2 - ay^2} = \alpha + \beta\sqrt{b}$ for some $\alpha, \beta \in K$. Squaring both sides, we get

$$x^2 - ay^2 = \alpha^2 + 2\alpha\beta\sqrt{b} + \beta^2 b \in K.$$

However $2\alpha\beta\sqrt{b} \notin K$, so either $\alpha = 0$ or $\beta = 0$.

If $\alpha = 0$, then

$$b\sigma(b) = x^2 - ay^2 = \beta^2(x + y\sqrt{a}) = \beta^2 b \in K.$$

Thus $\sigma(b) = \beta^2$. Since $\sigma^2 = \mathrm{Id}$, we have $\sigma(\beta)^2 = b$. But $b \in K^\times \setminus K^{\times 2}$. ↯ Thus we must have $\beta = 0$. So $\sqrt{x^2 - ay^2} = \alpha^2 \in K$. Hence $x^2 - ay^2 \in K^{\times 2}$.

($\Leftarrow$) Assume that $b\sigma(b) = x^2 - ay^2 \in K^{\times 2}$, i.e., $\gamma^2 = b\sigma(b)$ for some $\gamma \in L$. Let f be the minimal polynomial of $\sqrt{b}$ over F. To show $L = K(\sqrt{b})$ is normal, it suffices to show that all roots of f are in L.

The roots of f are $\sqrt{b}, -\sqrt{b}, \sqrt{\sigma(b)}$ and $-\sqrt{\sigma(b)}$. We already know that $\sqrt{b}$ (and hence $-\sqrt{b}$) are in L. As $b\sigma(b) = \gamma^2$, we have $\sqrt{b\sigma(b)} = \sqrt{b}\sqrt{\sigma(b)} = \pm\gamma \in L$. Therefore $\sqrt{\sigma(b)} = \gamma/\sqrt{b} \in L$. Thus $\sqrt{\sigma(b)} \in L$ and so $-\sqrt{\sigma(b)} \in L$. Hence L/F is normal.

Note that f is separable because it has no double roots (the four roots are distinct).[27] Thus L/F is Galois as desired. □

[27] Here we need $\operatorname{char} F \neq 2$. Otherwise $\sigma b = b$ and so $\sqrt{\sigma(b)} = \sqrt{b}$.

Algebra Exam

Summer 2016

Problem 4.9.1

Let V be a 3-dimensional $\mathbb{Q}$-vector space, and let $T : V \to V$ be a linear operator that has eigenvalues 1 and 2 but is not diagonalizable.

(a) What are the possible rational canonical forms of T?

(b) What are the possible Jordan canonical forms of the operator $\mathrm{Id} \otimes T : \mathbb{C} \otimes_{\mathbb{Q}} V \to \mathbb{C} \otimes_{\mathbb{Q}} V$ on the complexification?

Notes and Comments

Proof. This problem is an exact duplicate of Fall 2013 Problem 1. □

Problem 4.9.2

Let G be a finite p-group for a prime p, and let $H \lhd G$ be a nontrivial normal subgroup.

(a) Show that $H \cap Z(G) \neq \{1\}$, where $Z(G)$ is the center of G.

(b) Show that the hypothesis that H is normal in G cannot be omitted above.

Notes and Comments

Proof of (a). Recall that G acts on a normal subgroup H by conjugation, $g.h := ghg^{-1}$.[28] Consider H as a G-set. Then we have

$$\begin{aligned} H^G &= \{h \in H \,:\, g.h = h \text{ for all } g \in G\} \\ &= \{h \in H \,:\, ghg^{-1} = h \text{ for all } g \in G\} \\ &= \{h \in H \,:\, gh = hg \text{ for all } g \in G\} \\ &= H \cap Z(G). \end{aligned}$$

Since G is a finite p-group, the Class Equation says that

$$|H \cap Z(G)| = |H^G| \equiv |H| \pmod{p}.[29]$$

Since H is nontrivial, $|H| \equiv 0 \pmod{p}$. Thus $p \mid |H \cap Z(G)|$. That is, $H \cap Z(G)$ has order at least p. □

Proof of (b). We want a finite p-group G with a nontrivial non-normal subgroup H which intersects $Z(G)$ trivially.[30] The smallest such group is the dihedral group of order 8:

$$G := D_4 = \langle a, b \,:\, a^4 = 1 = b^2, ab = ba^{-1} \rangle$$

with the non-normal subgroup $H = \{1, b\}$. We have $a.b = aba^{-1} = a^2 b \notin H$, so $|H \cap Z(G)| = 1$. □

[28] That is, since $H \lhd G$, we have $ghg^{-1} \in H$ for all $h \in H, g \in G$. Moreover, it satisfies the axioms for a group action: $1.h = h$ and $g_1.(g_2.h) = g_1.(g_2 h g_2^{-1}) = g_1 g_2 h g_2^{-1} g_1^{-1} = (g_1 g_2).h$ for all $g_1, g_2 \in G, h \in H$.

[29] In general, if X is a G-set, then we can write $X = \mathcal{O}_1 \sqcup \mathcal{O}_2 \sqcup \cdots \sqcup \mathcal{O}_s \sqcup X^G$, for $\{\mathcal{O}_i\}_{i=1}^s$ the set of nontrivial orbits of the G action. Since each $\mathcal{O}_i$ is transitive as a G-set, it is isomorphic as a G-set to G/A_i for some proper subgroup A_i of G. Since $p \mid |G/A_i|$ for a p-group, this version of the Class Equation follows.

[30] In particular, such a group must be nonabelian. As there are no nonabelian groups of order p or p^2 for any prime p, we must look for a nonabelian group of order at least p^3. The easiest such group to write down will probably be good enough.

Problem 4.9.3

Let A and B be finitely generated abelian groups. Let $x \in A$ and $y \in B$ be elements of infinite order. Prove that $x \otimes y$ is a non-zero element of $A \otimes_{\mathbb{Z}} B$.

Notes and Comments

Proof. A finitely generated abelian group is the direct sum of a free abelian group and a torsion group. In particular, we say $A = \mathbb{Z}^e \oplus A_t$ and $B = \mathbb{Z}^f \oplus B_t$, where A_t, B_t are torsion groups.

Let $x \in A$ have infinite order and write $x = (m_1, \ldots, m_e, x_t)$, with each $m_i \in \mathbb{Z}$ and $x_t \in A_t$. As $x \in A$ has infinite order, it is not in the torsion subgroup of A, so $(m_1, \ldots, m_e) \neq 0$. Without loss of generality, $m_1 \neq 0$. Similarly, for $y \in B$ with infinite order, we can write $y = (n_1, \ldots, n_f, y_t)$ with $n_1 \neq 0$.

Define the projection maps $p\colon A \to \mathbb{Z}$ and $q\colon B \to \mathbb{Z}$ onto the first $\mathbb{Z}$ summand, so $p(x) = m_1$ and $q(y) = n_1$. Then we obtain a map $p \otimes q\colon A \otimes_{\mathbb{Z}} B \to Z \otimes_{\mathbb{Z}} Z$ such that $(p \otimes q)(x \otimes y) = m_1 \otimes n_1$. But $m_1 \otimes n_1 = m_1 n_1 \cdot (1 \otimes 1)$, which is non-zero because $m_1 n_1 \neq 0$ and $(1 \otimes 1)$ is a basis of $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$. Thus $(p \otimes q)(x \otimes y) \neq 0$ and so $x \otimes y \neq 0$. □

Problem 4.9.4

Let k be a field, and $k[x, y]$ a polynomial ring in two variables with coefficients in k.

(a) **The polynomials $y^2 - (x^3 - x)$ and $y^2 - x^3$ are both irreducible in $k[x, y]$. Pick one of these polynomials and prove that it is irreducible. In parts (b) and (c), you may assume both polynomials are irreducible in $k[x, y]$.**

(b) **Show that the quotient ring $k[x, y]/(y^2 - x^3)$ is a Noetherian integral domain.**

(c) **Show that the principal ideal $(y^2 - x^3)$ is not maximal in $k[x, y]$, but $(y^2 - x^3)$ is a maximal ideal in $k(x)[y]$.**

Notes and Comments

Proof of (a). Let $F(x, y) = y^2 - (x^3 - x)$ and $G(x, y) = y^2 - x^3$. Consider them as elements of $k[x][y]$.[31]

F is irreducible: This is an application of Eisenstein's Criterion.[32]. Indeed, we know that x is irreducible in $k[x]$. Furthermore:

- The leading coefficient of F (as a polynomial in y) is 1. Thus it is not divisible by x.
- The remaining coefficients of F are divisible by x.
- The constant term is $x^3 - x$, which is not divisible by x^2.

We conclude by Eisenstein's Criterion that F is irreducible in $k(x)[y]$.[33] We know that F is primitive as a polynomial over $k[x]$ (because its leading coefficient is 1) and so, by Gauss's Lemma, F is irreducible in $k[x][y]$.

[31] That is, consider F and G as polynomials in one variable y with coefficients from $k[x]$.

[32] Eisenstein's Criterion is always the first way you should try to prove a polynomial is irreducible because it is easy.

[33] Be careful! If you need to know that a polynomial is irreducible in $A[y]$ but A is not a field, then Eisenstein's Criterion is not quite enough. Eisenstein's Criterion would just tell you that the polynomial is irreducible in $F[y]$, where F is the field of fractions of A.

G **is irreducible:** We can also prove that G is irreducible in $k[x,y]$, using an argument that would apply for F as well. Write $G(x,y) = y^2 - g(x)$ for $g(x) \in k[x]$ that is not a perfect square. Suppose G factors into two non-constant (with respect to y) polynomials,

$$G(x,y) = y^2 - g(x) = (y - p(x))(y - q(x))$$

for $p(x), q(x) \in k[x]$. Expanding this yields

$$y^2 - g(x) = y^2 - (p(x) + q(x))y + p(x)\,q(x).$$

The coefficient of y is 0, so $p(x) + q(x) = 0$, so $q(x) = -p(x)$. Now we get

$$y^2 - g(x) = y^2 - p(x)^2.$$

So $g(x) = p(x)^2$, contradicting the fact that $g(x)$ is not a perfect square. Therefore $G = y^2 - g(x)$ is not a product of non-constant (with respect to y) polynomials. Hence G is irreducible in $k(x)[y]$. Now, as before, G is primitive and Gauss's Lemma tells us that G is irreducible in $k[x][y]$. □

Proof of (b). By the Hilbert Basis Theorem, $k[x,y]$ is Noetherian. Any quotient of a Noetherian ring is Noetherian[34], so $k[x,y]/(y^2 - x^3)$ is Noetherian.

Since $y^2 - x^3$ is irreducible and $k[x,y]$ is a UFD, $y^2 - x^3$ is prime. Thus $(y^2 - x^3)$ is a prime ideal and so $k[x,y]/(y^2 - x^3)$ is a domain. □

Proof of (c). Here are some proper ideals of $k[x,y]$ that properly contain $(y^2 - x^3)$:

$$(x,y),\ (x,y^2),\ (x^2,y),\ (x^2,y^2),\ (x^3,y),\ (x^3,y^2).$$[35]

Thus $(y^2 - x^3)$ is not maximal in $k[x,y]$.

Now we show that $(y^2 - x^3)$ is a maximal ideal in $k(x)[y]$. We know $y^2 - x^3$ is irreducible in $k[x][y]$ from part (a). So, by Gauss's Lemma, it is irreducible in $k(x)[y]$. This means that $(y^2 - x^3)$ is maximal among proper principal ideals of $k(x)[y]$.[36] Since $k(x)[y]$ is a PID (because $k(x)$ is a field), $(y^2 - x^3)$ is maximal among *all* proper ideals. □

Problem 4.9.5

Consider a tower of fields $K \subset F \subset E$.

(a) Show that the extension E/K is algebraic if and only if E/F and F/K are algebraic.

(b) Suppose that E/K is an algebraic extension of fields, and F/K is any extension of K (and E, F lie in some common field). Show that the extension EF/F is algebraic.

Notes and Comments

Proof. This problem is an exact duplicate of Summer 2013 Problem 4. □

[34] This fact is proved on the problem 6(b) of the 2014 summer algebra (8.5.6). However, in the context of this problem, just assume it.

[35] Notice that none of these is a principal ideal. Indeed, no proper principal ideal can properly contain $(y^2 - x^3)$ (see the next footnote).

[36] That is, no proper principal ideal properly contains (π) if π is irreducible. Indeed, if $(d) \supseteq (\pi)$, then $d \mid \pi$. So either d is a unit (meaning (d) is the whole ring) or d is associate to π (meaning $(d) = (\pi)$).

Problem 4.9.6

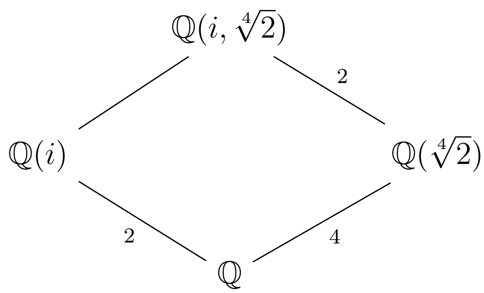
Let L be the splitting field of $x^4 - 2$ over $\mathbb{Q}$.

(a) Determine the degree $[L : \mathbb{Q}]$, and generators for $\mathrm{Gal}(L/\mathbb{Q})$. Also determine the isomorphism class of this Galois group (e.g., a standard group like $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$).

(b) Write down the complete lattice of subgroups of the Galois group, and the corresponding lattice of intermediate fields between L and $\mathbb{Q}$. Identify a majority of the intermediate fields, both as an appropriate fixed field, and with generator(s) over $\mathbb{Q}$.

Notes and Comments

Proof of (a). Since the 4^{th} roots of unity are $\{1, i, -1, -i\}$, the roots of $x^4 - 2$ are $\{\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}\}$. Consider the following lattice of extensions:



Since $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt[4]{2})$ are simple extensions, it is easy to see that $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$. The extension $\mathbb{Q}(i, \sqrt[4]{2})$ definitely contains the splitting field for the polynomial since all the roots are in this field. The key fact to notice here is that, since $\mathbb{Q}(\sqrt[4]{2}) \subset \mathbb{R}$, it cannot contain the complex roots of the polynomial.

Since $[\mathbb{Q}(i) : \mathbb{Q}] = 2$, it follows that $[\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}(\sqrt[4]{2})] \leq 2$ and, since $i \notin \mathbb{Q}(\sqrt[4]{2})$, we know $[\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}(\sqrt[4]{2})] \neq 1$. Thus the index is 2.

We know that $\mathbb{Q}(\sqrt[4]{2}) \subsetneq L \subset \mathbb{Q}(i, \sqrt[4]{2})$. Furthermore, $[\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}(\sqrt[4]{2})] = 2$ tells us there cannot be another field between these two. Hence $L = \mathbb{Q}(i, \sqrt[4]{2})$ is the splitting field of $x^4 - 2$ and

$$[L : \mathbb{Q}] = [\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}(\sqrt[4]{2})][\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 8.$$

That is, $\boxed{[L : \mathbb{Q}] = 8}$.

Because L is the splitting field of an irreducible polynomial over a field of characteristic zero, we know that the extension $L/\mathbb{Q}$ is Galois. So $|\mathrm{Gal}(L/\mathbb{Q})| = [L : \mathbb{Q}] = 8$. To determine the structure of the group, first notice that $\mathbb{Q} \subset \mathbb{Q}(\sqrt[4]{2}) \subset L$ and $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not normal.[37] By the Galois Correspondence, this extension corresponds to a subgroup of $\mathrm{Gal}(L/\mathbb{Q})$ which is not normal and so $\mathrm{Gal}(L/\mathbb{Q})$ cannot be abelian. There are only two nonabelian groups of order 8, namely D_4 and the quaternions.

One way to see that $\mathrm{Gal}(L/\mathbb{Q})$ is not isomorphic to the quaternions is that the quaternion group has only one subgroup of order 2. However, $[L : \mathbb{Q}(\sqrt[4]{2})] = [L : \mathbb{Q}(i\sqrt[4]{2})] = 2$ and $\mathbb{Q}(\sqrt[4]{2}) \neq \mathbb{Q}(i\sqrt[4]{2})$ (one is purely real) and so $\mathrm{Gal}(L/\mathbb{Q})$ has at least two subgroups of order 2. Thus, $\boxed{\mathrm{Gal}(L/\mathbb{Q}) \cong D_4}$.[38]

[37] Indeed, $\mathbb{Q}(\sqrt[4]{2})$ contains $\sqrt[4]{2}$ but not all of the roots of its minimal polynomial over $\mathbb{Q}$.

[38] Within the time constraints of the actual qual, we thought "I bet it's going to be D_4" and then found elements that would generate D_4 in the group via a little guess and check and intuition. We're adding some more concrete ways to know it's D_4 in case you find "follow your nose" to be rather unhelpful and/or obnoxious advice.

Recall that to define automorphisms in $\mathrm{Gal}(L/\mathbb{Q})$, we need only consider what these automorphisms do to i and $\sqrt[4]{2}$ (since the base field must be fixed). Also, remember that roots x^4-2 must map to other roots of the polynomial under the automorphisms. We can define automorphisms by how they act on the generators of the extension, so we take τ to be the automorphism such that $\tau(i)=-i$ and $\tau(\sqrt[4]{2})=\sqrt[4]{2}$ and take σ such that $\sigma(i)=1$ and $\sigma(\sqrt[4]{2})=i\sqrt[4]{2}$. This gives us the following table of group elements:

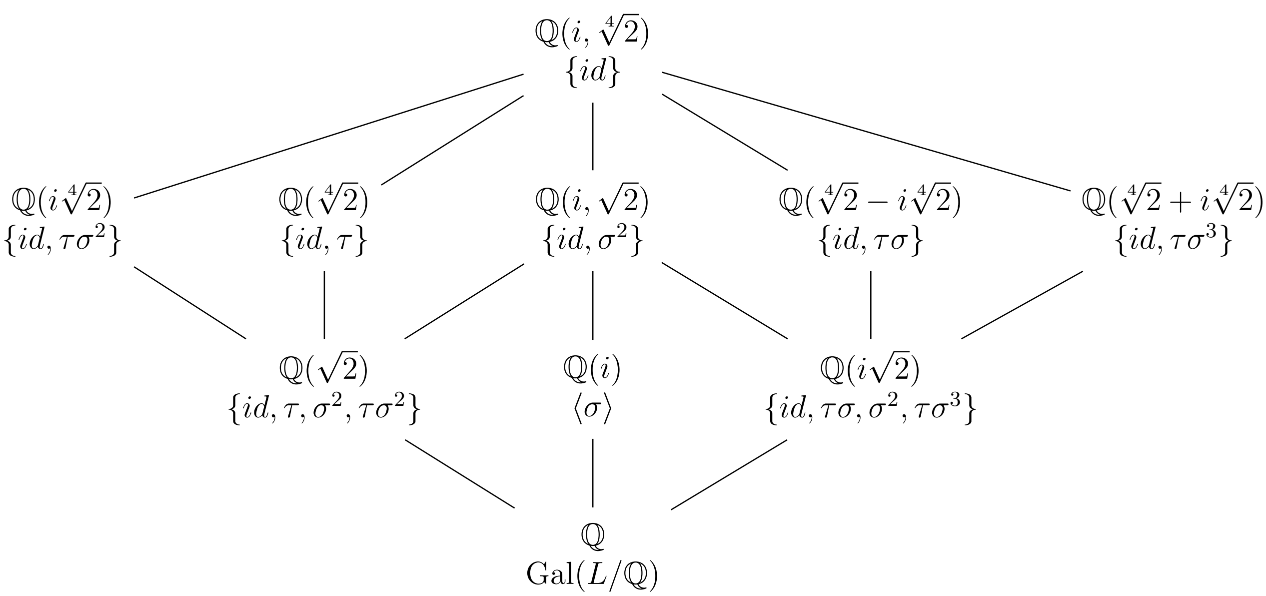
	id	σ	σ^2	σ^3	τ	$\tau\sigma$	$\tau\sigma^2$	$\tau\sigma^3$
i	i	i	i	i	$-i$	$-i$	$-i$	$-i$
$\sqrt[4]{2}$	$\sqrt[4]{2}$	$i\sqrt[4]{2}$	$-\sqrt[4]{2}$	$-i\sqrt[4]{2}$	$\sqrt[4]{2}$	$-i\sqrt[4]{2}$	$-\sqrt[4]{2}$	$i\sqrt[4]{2}$

We can now check that $\mathrm{Gal}(L/\mathbb{Q})$ satisfies the presentation for $D_4=\langle\sigma,\tau:\sigma^4=\mathrm{id},\tau^2=\mathrm{id},\sigma\tau\sigma=\tau\rangle$. $\square$

Proof of (b). The lattice of subgroups and corresponding subfields is below. Most of it is straightforward, but we think the reason this question asked for a "majority of the intermediate fields" is that two of the subfields, $\mathbb{Q}(\sqrt[4]{2}-i\sqrt[4]{2})$ and $\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$, are pretty nonobvious.

To see that these are each extensions of degree 4 over $\mathbb{Q}$, you can show that $\sqrt[4]{2}\pm i\sqrt[4]{2}$ are not roots of any quadratic polynomials over $\mathbb{Q}$ but they are roots of quartic polynomials, so they are not contained in any degree 2 extensions but are contained in degree 4 extensions. To see that $\mathbb{Q}(\sqrt[4]{2}-i\sqrt[4]{2})\neq\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$, we can use the Galois Correspondence as follows:

- Each subfield is a degree 4 extension, so each corresponds to a subgroup of $\mathrm{Gal}(L/\mathbb{Q})$ of order 2 which fixes the field.
- Note that $\sqrt[4]{2}-i\sqrt[4]{2}$ is fixed by $\tau\sigma$, while $\sqrt[4]{2}+i\sqrt[4]{2}$ is not. So $\sqrt[4]{2}-i\sqrt[4]{2}$ is in the degree 4 fixed field corresponding to $\{\mathrm{id},\tau\sigma\}$ and $\sqrt[4]{2}+i\sqrt[4]{2}$ is not.
- Therefore $\mathbb{Q}(\sqrt[4]{2}-i\sqrt[4]{2})\neq\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$ and $\mathbb{Q}(\sqrt[4]{2}-i\sqrt[4]{2})$ corresponds to $\{\mathrm{id},\tau\sigma\}$. Similarly, we find that $\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$ corresponds to $\{\mathrm{id},\tau\sigma^3\}$.



$\square$

Algebra Exam

Fall 2016

Problem 4.10.1

A linear operator E on a finite-dimensional k-vector space V is a *projection* if V admits a direct sum decomposition $V = U \oplus W$ where U and W are E-invariant subspaces such that $E|_U = \mathrm{Id}$ and $E|_W = 0$.

(a) Let $E, F \in \mathrm{End}_k(V)$ be projections. Show that, if E and F commute, EF is a projection.

(b) Is the converse true? That is, if E and F are projections such that EF is also a projection, must E and F commute?

Notes and Comments

Proof of (a). Since E is a projection, V admits a direct sum decomposition $V = U_E \oplus W_E$ such that $Eu = u$ for $u \in U_E$ and $Ew = 0$ for $w \in W_E$. Thus, if $(b_1, \ldots, b_k)$ is a basis of U_E and $(b_{k+1}, \ldots, b_n)$ is a basis of W_E, then $(b_1, \ldots, b_n)$ is a basis of V and, with respect to this basis, E can be written as a block-diagonal matrix in the following way:

$$[E] = \begin{pmatrix} I & \\ & 0 \end{pmatrix}$$

where I denotes the k-by-k identity matrix and 0 denotes the $(n-k)$-by-$(n-k)$ zero matrix. This shows that E is diagonalizable, with diagonal matrix consisting of 1s and 0s. Clearly the same holds for F, in some basis.

Because E and F are diagonalizable and they commute, they are *simultaneously* diagonalizable, say in basis $(v_1, \ldots, v_n)$. In this basis, E is a diagonal matrix consisting of 1s and 0s in some order, and the same holds for F in this basis. Let's write $[E] = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ and $[F] = \mathrm{diag}(\varepsilon'_1, \ldots, \varepsilon'_n)$, where each ε_i and each ε'_i is either 0 or 1. Then $[EF] = \mathrm{diag}(\varepsilon_1\varepsilon'_1, \ldots, \varepsilon_n\varepsilon'_n)$, so $[EF]$ is also a diagonal matrix consisting of 1s and 0s.

Let $U = \mathrm{span}\,\{v_i \mid \varepsilon_i\varepsilon'_i = 1\}$ and $W = \mathrm{span}\,\{v_i \mid \varepsilon_i\varepsilon'_i = 0\}$. Then $V = U \oplus W$, $EF|_U = \mathrm{Id}$, and $EF|_W = 0$. (Hence U and W are EF-invariant subspaces.) Thus EF is a projection as desired. $\square$

Proof of (b). $\boxed{\text{No}}$. For example, set $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and set $F = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. These are both projections:

- E is the identity on the span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and E is zero on the span of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$;
- F is the identity on the span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and F is zero on the span of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

However $EF = F$ and $FE = E$, showing that E and F do not commute although EF and FE are both projections. $\square$

Problem 4.10.2

Consider the group $\mathrm{SL}_2(\mathbb{F}_3)$ of 2-by-2 matrices of determinant 1 over the 3-element field $\mathbb{F}_3$.

(a) Determine the order of $\mathrm{SL}_2(\mathbb{F}_3)$.

(b) How many Sylow 3-subgroups does $\mathrm{SL}_2(\mathbb{F}_3)$ have? What is an example? What is the structure of its normalizer?

Notes and Comments

Proof of (a). First we count the elements of $\mathrm{GL}_2(\mathbb{F}_3)$ — the invertible matrices over $\mathbb{F}_3$. Each column of a matrix in $\mathrm{GL}_2(\mathbb{F}_3)$ is a vector in $(\mathbb{F}_3)^2$. There are nine vectors in $(\mathbb{F}_3)^2$ and any of them, except $(0,0)$, can be the first column of an invertible matrix. So there are eight options for the first column. For the second column, we can choose any of the nine vectors except the three multiples of the first column, so there are six options for the second column. Therefore $|\mathrm{GL}_2(\mathbb{F}_3)| = 8 \cdot 6 = 48$.

Recall that $\mathrm{SL}_2(\mathbb{F}_3)$ is the kernel of the (surjective) homomorphism $\det\colon \mathrm{GL}_2(\mathbb{F}_3) \to \mathbb{F}_3^\times$. As a result, $\mathrm{GL}_2(\mathbb{F}_3)/\mathrm{SL}_2(\mathbb{F}_3)$ is isomorphic to $\mathbb{F}_3^\times$. So $|\mathrm{GL}_2(\mathbb{F}_3)/\mathrm{SL}_2(\mathbb{F}_3)| = 2$ and thus $\boxed{|\mathrm{SL}_2(\mathbb{F}_3)| = |\mathrm{GL}_2(\mathbb{F}_3)|/2 = 24}$. □

Proof of (b). Let n_3 be the number of Sylow 3-subgroups. By the Sylow Theorem, $n_3 \mid 8$ and $n_3 \equiv 1 \mod 3$. These conditions require that n_3 be either 1 or 4. Perhaps the easiest way to see that $n_3 \neq 1$ is by finding more than one Sylow 3-subgroup: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ generate two different subgroups of $\mathrm{SL}_2(\mathbb{F}_3)$, each of order 3. Thus $n_3 \neq 1$, so $\boxed{n_3 = 4}$. Thus there are four Sylow 3-subgroups. (We have already shown two examples of such subgroups.)

The Sylow Theorem also tells us that any two Sylow 3-subgroups are conjugate, so in particular their normalizers will be isomorphic. Thus it suffices to describe the normalizer of just one Sylow 3-subgroup, say $H = \left\{1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right\}$.

By the Sylow Theorem, the normalizer $N(H)$ has index $n_3 = 4$ and so its order is 6. By inspection, we determine that $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ generates a cyclic subgroup U of order 6 that contains H. In fact, U is the subgroup of all upper-triangular matrices in $\mathrm{SL}_2(\mathbb{F}_3)$. We know H is normal in U because $[U : H] = 2$. Therefore U is the normalizer of H and it is cyclic of order 6. □

Problem 4.10.3

Let k be a field, let V be a finite-dimensional k-vector space, and let $T : V \to V$ be a linear operator on V. Show that V admits a direct sum decomposition $V = U \oplus W$, where U and W are subspaces satisfying

(a) $T(U) \subseteq U$ and $T(W) \subseteq W$.

(b) The restriction $T|_U : U \to U$ is nilpotent.

(c) The restriction $T|_W : W \to W$ is invertible.

Notes and Comments

Proof. Consider V as a $k[x]$-module. Since k is a field, $k[x]$ is a PID. Thus V has a primary decomposition $V = U \oplus W$ where U is the x-primary component of V and W is the sum of all other primary components. That is, U and W are T-invariant subspaces and U is annihilated by x^n (when $n = \dim V$). Thus $T|_U$ is nilpotent.

Finally, suppose $T(w) = 0$ for some $w \in W$. Then $\langle w \rangle$, the 1-dimensional subspace generated by w, is annihilated by x. Hence $w \in U$. Thus $w \in W \cap U = \{0\}$. Hence $T|_W$ is injective. As V is finite-dimensional, this means that $T|_W$ is an isomorphism (hence invertible). □

Problem 4.10.4

Unique factorization.

(a) Show that $\mathbb{Z}[\sqrt{-5}]$ is a Noetherian integral domain, but not a UFD.

(b) Show that Noetherian integral domain in which every irreducible element is a prime element is a UFD.

Notes and Comments

Proof of (a). We first note that $\mathbb{Z}[\sqrt{-5}]^\times = \{\pm 1\}$. Since

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \cdot 3$$

and no factor is an associate of another, we know that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

To show that $\mathbb{Z}[\sqrt{-5}]$ is a Noetherian integral domain, note that $\mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/\langle x^2 + 5 \rangle$. Since $\mathbb{Z}$ is a UFD, so is $\mathbb{Z}[x]$. Hence all irreducibles are prime and so $x^2 + 5$ is prime in $\mathbb{Z}[x]$. Thus $\mathbb{Z}[\sqrt{-5}]$ can be realized as the quotient of a ring by a prime ideal and is thus an integral domain.

Furthermore, $\mathbb{Z}[x]$ is Noetherian by the Hilbert Basis Theorem. As the quotient of a Noetherian ring is Noetherian, $\mathbb{Z}[\sqrt{-5}]$ is Noetherian. □

Proof of (b). Let A be a Noetherian integral domain. Then there is factorization into irreducibles in A.[39] Assume all irreducibles in A are prime.

Let $a \in A$ with $a \neq 0$ and $a \notin A^\times$. Suppose we have two factorizations of a:

$$a = \pi_1 \cdots \pi_r = \pi'_1 \cdots \pi'_s$$

where π_i, π'_j are irreducibles in A and $r \leq s$. We will proceed by induction on r.

Base case: If $r = 1$, then π_1 is being split into irreducible factors. That is, $s = 1$ and π'_1 is an associate of π_1.

Induction: If $r > 1$, then we know that π_1 divides the second factorization. Since π_1 is also prime (by assumption), π_1 divides one of the irreducibles in the second factorization. Without loss of generality, $\pi_1 \mid \pi'_1$. Then $\pi'_1 = u\pi_1$ where $u \in A^\times$. Dividing by π_1 (which is reasonable in an integral domain), we have

$$\pi_2 \cdots \pi_r = u\pi'_2 \cdots \pi'_s.$$

By the inductive hypothesis, $r = s$ and the remaining factors are associates of one another. Hence, up to unit multiples, a has a unique factorization as desired. □

[39] This is a worthwhile problem on its own and, in fact, made an appearance on the Summer 2014 exam.

Problem 4.10.5

Let L be the splitting field of $x^9 - 5^3$ over $\mathbb{Q}$ and ζ_9 a primitive 9th root of unity in $\mathbb{C}$.

(a) Determine the degree of $[L : \mathbb{Q}]$, giving reasons to support your statements.

(b) Determine the isomorphism classes of $\mathrm{Gal}(L/\mathbb{Q}(\zeta_9))$ and $\mathrm{Gal}(L/\mathbb{Q}(\sqrt[3]{5}))$.

(c) Show that $G = \mathrm{Gal}(L/\mathbb{Q})$ has a normal subgroup H with $G/H \cong S_3$, the symmetric group on 3 letters.

Notes and Comments

Proof of (a). The degree of $[L : \mathbb{Q}]$ is 18. The reasoning follows precisely from problem 2 on the Summer 2012 exam (4.1.2). □

Proof of (b). We claim that $\mathrm{Gal}(L/\mathbb{Q}(\zeta_9)) = \mathbb{Z}/3\mathbb{Z}$ and $\mathrm{Gal}(L/\mathbb{Q}(\sqrt[3]{5})) = \mathbb{Z}/6\mathbb{Z}$. Since there is an isomorphism of Galois groups (Theorem 1.12 in Lang) this follows immediately from noticing that $\mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q}) = (\mathbb{Z}/9\mathbb{Z})^\times = \mathbb{Z}/6\mathbb{Z}$ and $|\mathrm{Gal}(L/\mathbb{Q}(\zeta_9))| = 3$. Alternatively, we could write $G = \mathrm{Gal}(L/\mathbb{Q})$ as a semidirect product and realize that these are the component quotients. □

Proof of (c). The subextension $\mathbb{Q}(\zeta_3, \sqrt[3]{5})\mathbb{Q}$ is the splitting field of $x^3 - 5$ which is irreducible and has Galois group S_3. One way to see that this is the Galois group is to note that it has order 6 and cannot be abelian since $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$ is not normal. Thus, by the Fundamental Theorem of Galois Theory, we can find the desired subgroup H corresponding to the splitting field of $\Phi_9(x) = x^6 + x^3 + 1$ over $\mathbb{Q}$. □

Problem 4.10.6

Let $f \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 3$, and let K be the splitting field of f over $\mathbb{Q}$. Suppose that $\mathrm{Gal}(K/\mathbb{Q}) \cong S_n$, the symmetric group.

(a) Show that f is irreducible.

(b) If α is a root of f in K, show that $\mathrm{Aut}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is trivial, that is show that every automorphism of $\mathbb{Q}(\alpha)$ which fixes $\mathbb{Q}$ pointwise is the identity.

(c) If $n \geq 4$, show that $\alpha^n \notin \mathbb{Q}$.

Notes and Comments

Proof of (a). In order to obtain a contradiction, assume that f is not irreducible. So $f = gh$ with both $0 < \deg(g), \deg(h) < n$. Then $\deg(g) + \deg(h) = n$ and the maximum possible degree of $K/\mathbb{Q}$ is $\deg(g)! \cdot \deg(h)! < n!$. ↯ Hence f is irreducible. □

Proof of (b). Let $\sigma \in \mathrm{Aut}(\mathbb{Q}(\alpha)/\mathbb{Q})$. Then $\sigma(\alpha) \in \mathbb{Q}(\alpha)$ must be a root of f since f is irreducible. However, if $\mathbb{Q}(\alpha)$ contains a root of f other than α, then $[K : \mathbb{Q}(\alpha)] < (n-2)!$. As $[K : \mathbb{Q}] = n!$, it must be that $\sigma(\alpha) = \alpha$. Hence $\sigma = \mathrm{Id}$. □

Proof of (c). To the contrary, suppose that $\alpha^n \in \mathbb{Q}$. Let $k \in \mathbb{Z}$ be the smallest positive integer such that $\alpha^k \in \mathbb{Q}$. By part (a), we know that f is irreducible and so we must have $k = n$. Moreover, $f = c(x^n - \alpha^n)$ for some $c \in \mathbb{Q}^\times$. However, this would imply that the degree of the extension $L/\mathbb{Q}$ is at most $n\varphi(n) < n!$ since $n \geq 4$. ↯ Thus $\alpha^n \notin \mathbb{Q}$. □

Algebra Exam

Summer 2017

Problem 4.11.1

Let p be an odd prime. Let $O_2(\mathbb{F}_p) = \{A \in GL_2(\mathbb{F}_p) : AA^t = A^tA = I\}$. Then $O_2(\mathbb{F}_p) \leq GL_2$ is a subgroup called the *(standard) orthogonal group* in $GL_2(\mathbb{F}_p)$.

(a) Let $p = 5$. Show that $\#O_2(\mathbb{F}_p) = 8$ and classify $O_2(\mathbb{F}_p)$ up to isomorphism (i.e., give it a more familiar name).

(b) For p an arbitrary odd prime, show that $O_2(\mathbb{F}_p)$ always has a nontrivial normal subgroup of index 2 and write down the corresponding exact sequence of groups. Does this sequence split?

Notes and Comments

Proof of (a). This solution is rather computational.

If $A \in O_2(\mathbb{F}_5)$, then

$$\det(A)^2 = \det(A)\det(A^t) = \det(AA^t) = \det(I) = 1,$$

so $\det(A) = \pm 1$.[40]

Let $W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and note that $WW^t = I$, i.e., $W \in O_2(\mathbb{F}_5)$. For any $A \in SO_2(\mathbb{F}_5)$, we have $WA \in O_2(\mathbb{F}_5)$ with $\det(WA) = -1$. Thus $\#O_2(\mathbb{F}_5) = 2\#SO_2(\mathbb{F}_5)$. So it suffices to show $\#SO_2(\mathbb{F}_5) = 4$.

Let $A \in SO_2(\mathbb{F}_5)$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By definition, $A^t = A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Thus $\boxed{a = d}$ and $\boxed{b = -c}$, and so $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $1 = \det(A) = a^2 + b^2$.

Now we can find all the matrices in $SO_2(\mathbb{F}_5)$ by finding solutions to $a^2 + b^2 = 1$ over $\mathbb{F}_5$. Computing a table of squares, the only solutions are $\{(1,0),(0,1),(4,0),(0,4)\}$. Thus

$$SO_2(\mathbb{F}_5) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \right\},$$

and we see that $B = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$ generates this group. By our earlier argument, $O_2(\mathbb{F}_5) = \langle B, W \rangle$. Through another check, we see that this group is non-abelian. There is only one group of order 8 with a cyclic subgroup of order 4, namely D_8. Hence $\boxed{O_2(\mathbb{F}_5) \cong D_8}$. □

Proof of (b). Using W as in part (a), $SO_2(\mathbb{F}_p) \leq O_2(\mathbb{F}_p)$ has index 2. Since any subgroup of index 2 is normal, $SO_2(\mathbb{F}_p)$ is normal in $O_2(\mathbb{F}_p)$. Moreover, $SO_2(\mathbb{F}_p)$ is nontrivial because $a^2 + b^2 = 1$ always has $\{(1,0),(0,1)\}$ as solutions.

As the determinant is multiplicative, det is a homomorphism from $O_2(\mathbb{F}_p) \to \{\pm 1\}$ with kernel $SO_2(\mathbb{F}_p)$. This forms the exact sequence of groups

$$1 \longrightarrow SO_2(\mathbb{F}_p) \xrightarrow{\iota} O_2(\mathbb{F}_p) \xrightarrow{\det} \{\pm 1\} \longrightarrow 1 \quad .$$

[40] The polynomial $x^2 - 1$ has at most 2 roots, regardless of the field, and we have 2 solutions. Since 5 is odd, these solutions are distinct.

Consider $p = 5$ and suppose the sequence splits. Then $O_2(\mathbb{F}_5) = SO_2(\mathbb{F}_5) \times \{\pm 1\}$, the direct product of abelian groups. Hence $O_2(\mathbb{F}_5)$ is abelian, but we proved in part (b) that it is not. $\lightning$ So this sequence does not split in general.[41] $\square$

Problem 4.11.2

Let R be a commutative ring, let M be an R-module, and let $\phi : M \to M$ be an R-module homomorphism.

(a) Show that $\phi^2 = 0$ if and only if $\phi(M) \subseteq \ker\phi$.

(b) Suppose $R = F$ is a field and $M = V$ is finite dimensional as an F-vector space. Show that there is an ordered basis β for V such that $[\phi]_\beta$ has the block form $\begin{pmatrix} O & A \\ O & O \end{pmatrix}$, i.e., has zeros in all blocks except possibly the upper right-hand corner.

Notes and Comments

Proof of (a). Note that

$$\phi^2 = 0 \quad \Leftrightarrow \quad \phi(\phi(x)) = 0 \ \forall x \in M \quad \Leftrightarrow \quad \phi(M) \subseteq \ker\phi.$$

$\square$

Proof of (b). Assume $\phi^2 = 0$.[42] Write $M = \ker\phi \oplus N$, where N is a complement of $\ker\phi$, and let $\{v_1, \ldots, v_m\}$ be a basis for $\ker\phi$, $\{w_1, \ldots, w_n\}$ a basis for N.

Consider the ordered basis $\beta = \{v_1, \ldots, v_m, w_1, \ldots, w_n\}$. Since $\phi^2 = 0$ and $w_j \notin \ker\phi$, we know that $\phi(w_j) \neq 0$ but $\phi^2(w_j) = 0$. Hence $\phi(w_j)$ can be written as a linear combination of the v_i's. Since $v_i \in \ker\phi$, $\phi(v_i) = 0$. Thus the matrix for ϕ in the ordered basis β has the desired form.

To be more explicit, because that is loved by all, the column vectors (generically) appear as:

$$[\phi(v_i)]_\beta = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, [\phi(w_j)]_\beta = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Stringing these together, $[\phi]_\beta = \begin{bmatrix} O & A \\ O & O \end{bmatrix}$ where $A = [a_{ij}]$ with $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. $\square$

[41] While this sequence admits a section ($s : \{\pm 1\} \to O_2(\mathbb{F}_p)$ given by $1 \mapsto I$ and $-1 \mapsto W$), this does not imply that the short exact sequence splits in the category of groups.

[42] This is implicitly assumed. No, it isn't explicitly stated anywhere.

Problem 4.11.3

Let V, W be finite dimensional vector spaces over a field F, and let $\phi : V \to W$ be an F-linear map. Let $\phi^* : V^* \to W^*$ be the dual map. Show that ϕ is surjective if and only if ϕ^* is injective.

Notes and Comments

Proof. Assume that ϕ is surjective. Suppose $\phi^*(f_1) = \phi^*(f_2)$, where $f_1, f_2 \in V^*$. We want to show that $f_1 = f_2$. Let $w \in W$. Since ϕ is surjective, $w = \phi(v)$ for some $v \in V$. Now

$$f_1(w) = f_1 \circ \phi(v) = \phi^*(f_1) = \phi^*(f_2) = f_2 \circ \phi(v) = f_2(w).$$

As w was arbitrary, $f_1 = f_2$. Hence ϕ^* is injective.

Conversely, assume that ϕ^* is injective. Let $w \in W$ with $w \neq 0$ (trivially, $0 \in \phi(V)$). We want to show $\exists v \in V$ such that $\phi(v) = w$.

Extend $\{w\}$ to a basis β for W. Let β^* be the dual basis to β, where $f \in \beta^*$ such that $f(w) = 1$ and f vanishes on all other elements of β. So $f \neq 0$ and, by injectivity, $\phi^*(f) \neq 0$. Thus $\exists v \in V$ such that

$$\phi^*(f)(v) = f(\phi(v)) \neq 0.$$

By definition of f, $\phi(v) = cw$. In particular, by linearity, $\phi\left(\frac{1}{c}v\right) = w$. Hence ϕ is surjective. □

Problem 4.11.4

Polynomials

(a) Characterize (and determine the number of) all proper, non-trivial ideals of the quotient rings $\mathbb{Q}[x]/(x^4-1)$ and $\mathbb{Q}[x]/(x^4+1)$, where (f) denotes the ideal of $\mathbb{Q}[x]$ generated by f.

(b) Characterize the structure of the quotient ring $\mathbb{Q}[x,y]/(x^2+1, x^2+y^4)$ by showing it is isomorphic to something involving simple rings (in both senses of the word simple).

Notes and Comments

Solution for (a). The ideals of $\mathbb{Q}[x]/(x^4-1)$ correspond canonically to ideals of $\mathbb{Q}[x]$ that contain (x^4-1). Since $\mathbb{Q}[x]$ is a PID, every ideal that contains (x^4-1) is generated by an element of $\mathbb{Q}[x]$ that divides x^4-1. Now x^4-1 factors into irreducibles in $\mathbb{Q}[x]$ as $x^4-1 = (x-1)(x+1)(x^2+1)$, so the ideals that contain (x^4-1) are $(x-1)$, $(x+1)$, (x^2+1), $((x-1)(x+1))$, $((x-1)(x^2+1))$, and $((x+1)(x^2+1))$. By applying the quotient map, these ideals of $\mathbb{Q}[x]$ correspond to all the proper, non-trivial ideals in $\mathbb{Q}[x]/(x^4-1)$:

$$((x+1)(x^2+1)), ((x-1)(x^2+1)), ((x-1)(x+1)), (x^2+1), (x+1), (x-1).$$

Now let's consider x^4+1, which we claim is irreducible in $\mathbb{Q}[x]$. By Gauss' Lemma, it's equivalent to show that x^4+1 is irreducible in $\mathbb{Z}[x]$. It clearly has no linear factors (as there are no rational roots), and factoring into monic quadratics generates no solutions. Hence irreducibility follows.

Since x^4+1 is irreducible in $\mathbb{Q}[x]$ and $\mathbb{Q}[x]$ is a PID, (x^4+1) is a maximal ideal in $\mathbb{Q}[x]$. Thus $\mathbb{Q}[x]/(x^4+1)$ is a field, and so it has no proper, non-trivial ideals. □

Solution for (b). First of all, $x^2+y^4 = (y^4-1)+(x^2+1)$, so the two ideals (x^2+1, x^2+y^4) and (x^2+1, y^4-1) are the same. So the quotient ring in question is equal to $\mathbb{Q}[x,y]/(x^2+1, y^4-1)$.

First we will examine $\mathbb{Q}[x]/(x^2+1)$. Since x^2+1 is irreducible in $\mathbb{Q}[x]$ with root i, we have $\mathbb{Q}[x]/(x^2+1) \cong \mathbb{Q}(i)$. Thus we have

$$\mathbb{Q}[x,y]/(x^2+1, y^4-1) \cong (\mathbb{Q}[x]/(x^2+1))[y]/(y^4-1) \cong \mathbb{Q}(i)[y]/(y^4-1).$$

Now y^4-1 splits over the field $\mathbb{Q}(i)$, as $y^4-1 = (y-1)(y-i)(y+1)(y+i)$. Therefore, by the Chinese Remainder Theorem, we have

$$\mathbb{Q}[x,y]/(x^2+1, y^4-1) \cong \mathbb{Q}(i)[y]/(y-1) \times \mathbb{Q}(i)[y]/(y-i) \times \mathbb{Q}(i)[y]/(y+1) \times \mathbb{Q}(i)[y]/(y+i).$$

Each of the four factors is isomorphic to $\mathbb{Q}(i)$, so the whole thing is isomorphic to the ring $\mathbb{Q}(i)^4$.

The two senses of the word "simple" are the colloquial sense ($\mathbb{Q}(i)$ is not a complicated ring) and the technical sense ($\mathbb{Q}(i)$ is a ring with no proper, non-trivial ideals). □

Problem 4.11.5

Let E be the splitting field of x^5-3 over $\mathbb{Q}$, F the splitting field of x^5-7 over $\mathbb{Q}$, and put $L = EF$, their compositum. Note: You may assume that x^5-7 has no roots in E and x^5-3 has no roots in F.

(a) Determine the degree $[L : \mathbb{Q}]$.

(b) Determine the isomorphism class of $\mathrm{Gal}(L/F)$.

(c) Determine the number of Sylow p-subgroups for each prime p dividing the order of $\mathrm{Gal}(L/\mathbb{Q})$.

Notes and Comments

Proof of (a). By a standard argument, $E = \mathbb{Q}(\sqrt[5]{3}, \zeta_5)$ and $F = \mathbb{Q}(\sqrt[5]{7}, \zeta_5)$ where ζ_5 is a primitive 5th root of unity. Hence $[F : \mathbb{Q}] = 20$. Since F does not contain any roots of x^5-3, we obtain $L = EF$ by adjoining $\sqrt[5]{3}$ to F.

By multiplicativity of degrees in towers,

$$[L : \mathbb{Q}] = [L : F][F : \mathbb{Q}] = 5 \cdot 20 = 100.$$

□

Proof of (b). Since $L = F(\sqrt[5]{3})$, we know that $|\mathrm{Gal}(L/F)| = 5$ by the Galois correspondence. Hence $\mathrm{Gal}(L/F)$ is isomorphic to the cyclic group of order 5. □

Proof of (c). Let $G = \mathrm{Gal}(L/\mathbb{Q})$. Then $|G| = 100 = 2^2 \cdot 5^2$. Let n_p be the number of Sylow p-subgroups.

By the Sylow theorems, we know that $n_p \equiv 1 \mod p$ and that n_p divides $|G|/p^2$ (since all our primes occur in squares). Consequently, $\boxed{n_5 = 1}$ and $n_2 \in \{1, 5, 25\}$.

Note that G has a cyclic subgroup of order 4 generated by ϱ where ϱ fixes $\sqrt[5]{3}$ and $\sqrt[5]{7}$ and sends $\zeta_5 \mapsto \zeta_5^2$. Hence G has a cyclic Sylow 2-subgroup. By the Sylow theorem, all Sylow 2-subgroups are conjugate and hence cyclic of order 4.

Suppose $n_2 = 25$. Then G has 25 cyclic subgroups of order 4, each having two generators. Thus G has 50 distinct elements of order 4. Additionally, G has at least 25 elements of order 2.[43] Counting the Sylow 5-subgroup, we have at least 100 elements of prime power order. However, G must also contain elements of order 10, 20, and so on. So $|G| > 100$. ↯ Hence $\boxed{n_2 \neq 25}$.

Suppose $n_2 = 1$. Let P be the unique Sylow 5-subgroup and Q the unique Sylow 2-subgroup. Then $P, Q \lhd G$. Since P, Q are of prime-squared order, they are abelian. By normality and the fact that $P \cap Q = \{e\}$, P and Q commute. Hence $G = PQ$ is abelian, which we claim is false.

Consider $\sigma, \tau \in G$ given by

$$\sigma(\sqrt[5]{3}) = \sqrt[5]{3}\zeta_5, \sigma(\zeta_5) = \zeta_5^2$$

and

$$\tau(\sqrt[5]{(3)}) = \sqrt[5]{3}, \tau(\zeta_5) = \zeta_5^4.$$

Then $\sigma\tau(\sqrt[5]{3}) = \sqrt[5]{3}\zeta_5$ but $\tau\sigma(\sqrt[5]{3}) = \sqrt[5]{3}\zeta_5^4$. Hence $\sigma\tau \neq \tau\sigma$ and so G is not abelian. ↯ Thus $\boxed{n_2 \neq 1}$.

As $n_2 \in \{1, 5, 25\}$, we must have $\boxed{n_2 = 5}$. □

Problem 4.11.6

Let $i = \sqrt{-1} \in \mathbb{C}$, let $\zeta_5 \in \mathbb{C}$ be a primitive 5th root of unity, and put $E = \mathbb{Q}(\zeta_5)$.

(a) Show that $i \notin E$.

(b) Let $L = E(i)$. Consider the norm, $N_{L/E}$, from L to E. Show that the image of the norm consists of those elements in E which can be written as the sum of two squares in E.

(c) Determine the isomorphism class of $\mathrm{Gal}(L/\mathbb{Q})$.

(d) Determine whether a regular 20-gon is constructible by straightedge and compass.

Notes and Comments

Proof of (a). Let $F = \mathbb{Q}(\zeta_5 + \zeta_5^{-1})$ be the maximal real subfield of E. Then $[E : F] = 2$. Recall that $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/5\mathbb{Z})^\times \cong C_4$, the cyclic group of order 4. Since C_4 has only one subgroup of order 2, the Galois correspondence tells us that F is the only subfield of E with $[E : F] = 2$.

Now suppose $i \in E$. Then $\mathbb{Q} \subset \mathbb{Q}(i) \subset E$. By multiplicativity of degrees in towers,

$$4 = [E : \mathbb{Q}] = [E : \mathbb{Q}(i)][\mathbb{Q}(i) : \mathbb{Q}] = 2[E : \mathbb{Q}(i)],$$

and thus $[E : \mathbb{Q}(i)] = 2$. Thus $\mathbb{Q}(i) = F$ by the above. However, $\mathbb{Q}(i)$ is not a real subfield of E, and so $\mathbb{Q}(i) \neq F$. ↯ Hence $\boxed{i \notin E}$. □

Proof of (b). Every element of $L = E(i)$ can be written in the form $a + bi$ with $a, b \in E$. Since $i \notin E$, we have $[L : E] = 2$, so $i \mapsto -i$ defines the one non-trivial automorphism of L over E. Consequently, the norm is given by $N_{L/E}(a + bi) = (a + bi)(a - bi) = a^2 + b^2$. □

[43] Take $\sigma_{(m,n)}$ for $m, n \in \{1, \ldots, 5\}$ where $\sqrt[5]{3} \mapsto \sqrt[5]{3}\zeta_5^m$, $\sqrt[5]{7} \mapsto \sqrt[5]{7}\zeta_5^n$, and $\zeta_5 \mapsto \zeta_5^4$.

Proof of (c). We have $L = \mathbb{Q}(\zeta_5, i)$. Since $i = \zeta_4$ and $\operatorname{lcm}(4,5) = 20$, we have $L = \mathbb{Q}(\zeta_{20})$. Thus $\operatorname{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/20\mathbb{Z})^\times$ as groups. By the Chinese Remainder Theorem, $\mathbb{Z}/20\mathbb{Z} \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$ as rings. So

$$\operatorname{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/4\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times \cong \boxed{C_2 \times C_4},$$

the product of cyclic groups. □

Proof of (d). We prove that a regular 20-gon (or icosagon) can be constructed with straightedge and compass. This is the case if and only if $\cos(2\pi/20)$ is a constructible real number. Observe that

$$2\cos(2\pi/20) = e^{2\pi i/20} + e^{-2\pi i/20},$$

so $\cos(2\pi/20) \in \mathbb{Q}(\zeta_{20}) = L$. We have the tower of fields $\mathbb{Q} \subset \mathbb{Q}(\zeta_5 + \zeta_5{}^{-1}) \subset \mathbb{Q}(\zeta_5) \subset L$, and each field in this tower has degree 2 over the field it covers. This means that every real number in L is constructible, so $\cos(2\pi/20)$ is constructible. □

5

Analysis

Analysis Exam

Summer 2012

Problem 5.1.1

Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be a function. Give three criteria (ε/δ, open sets, sequences) for f to be continuous on A. Show that these definitions are equivalent.

Notes and Comments

Proof. The three equivalent versions of continuity are:

(1) (ε and δ): For every $a \in A$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \varepsilon$.

(2) (Sequential): For $a \in A$ and every sequence $\{x_n\}_{n=1}^{\infty} \subset A$ such that $\lim_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = f(a).$$

(3) (Open Sets): For every open $U \subset \mathbb{R}$, the set $f^{-1}(U)$ is open in A (with respect to the subspace topology).

We prove the equivalence by showing that the second and third conditions are equivalent to the first.

(1)⇔(2): Suppose that (1) holds and let $a \in A$ and $\{x_n\} \subset A$ such that $x_n \to a$.

Let $\varepsilon > 0$. By (1), there exists a δ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \varepsilon$. Choose $N \in \mathbb{N}$ such that, for $n \geq N$, we have $|x_n - a| < \delta$. Then $n \geq N$ also implies that $|f(x_n) - f(a)| < \varepsilon$, so $f(x_n) \to f(a)$ as $n \to \infty$.

To show that (2) implies (1) we argue by contraposition. Assume that there exists an $a \in A$ and $\varepsilon > 0$ such that, for all $\delta > 0$, there exists $x \in A$ such that $|x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$.

Thus we can take the sequence $\delta_n = \frac{1}{n}$ to obtain a sequence of $\{x_n\}$ with $|x_n - a| < \delta_n$ but $|f(x_n) - f(a)| \geq \varepsilon$. Clearly $x_n \to a$ but $f(x_n) \not\to f(a)$, proving the contrapositive.

(1)⇔(3): Assume that (1) holds and let $U \subseteq \mathbb{R}$ be an arbitrary open set. If $A \cap f^{-1}(U) = \varnothing$ we are done because $\varnothing$ is open.

Let $a \in f^{-1}(U)$. Then there exists $\varepsilon > 0$ such that $|y - f(a)| < \varepsilon$ implies that $y \in U$. Then, by (1), we can find a corresponding $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \varepsilon$. Hence $f(x) \in U$. Thus $B_\delta(a) \subseteq f^{-1}(U)$ and so a is an interior point of $f^{-1}(U)$. Since a was arbitrary, all points of $f^{-1}(U)$ are interior and the set is open in A.

Finally, assume that (3) is true and select an arbitrary $a \in A$ and $\varepsilon > 0$. Notice that $U = B_\varepsilon(f(a)) = \{y \in \mathbb{R} : |y - f(a)| < \varepsilon\}$ is open. By (3) we know that $f^{-1}(U)$ is open in A with $a \in f^{-1}(U)$. Since it is open, a is an interior point and there exists a $\delta > 0$ such that $|x - a| < \delta$ implies $x \in f^{-1}(U)$. This implies (1), completing our proof. □

Problem 5.1.2

Let Ω be an open connected subset of $\mathbb{C}$. Suppose the f_n is a sequence of holomorphic functions on Ω for each $n \geq 1$ and that the sequence $\{f_n\}$ converges to a function f uniformly on each compact subset of Ω.

(a) Show that f is holomorphic on Ω.

(b) Show that the sequence $\{f_n'\}$ of derivatives converges to f' uniformly on compact subsets of Ω.

Note: We say that g is holomorphic on Ω if $g'(z)$ exists for all $z \in \Omega$.

Notes and Comments

Proof of (a). Let T be a triangle contained in a disk of Ω. Then, for any n, we have $\int_{\partial T} f_n(z)dz = 0$ by Cauchy's Theorem. By uniform convergence on the disk in Ω,

$$\lim_{n\to\infty} \int_{\partial T} f_n(z)dz = \int_{\delta T} \lim_{n\to\infty} f_n(z)dz = \int_{\partial T} f_(z)dz = 0.$$

By Morera's Theorem, this is sufficient to show that f is holomorphic on the arbitrary disk; hence on Ω. □

Proof of (b). Let K be a compact subset of Ω. Let $\varepsilon > 0$ and consider $K' = \bigcup_{z\in K} \overline{D(z,\rho)} \subseteq \Omega$ where $K \subset K'$ and K' is compact. Then $f_n \overset{\text{unif.}}{\to} f$ on K'. Thus by definition of uniform convergence, $\exists N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \varepsilon\rho$ for all $n \geq N$ and $z \in K'$. Thus on each disk, $|f_n - f| < \varepsilon\rho$. Hence by Cauchy estimates, we have $|f_n' - f'(z)| < \varepsilon$. □

Problem 5.1.3

Let $C([0,1])$ be the complex vector space of continuous complex-valued functions on $[0,1]$.

(a) Suppose that $\{f_n\}$ is a sequence in $C([0,1])$ and that f is a function on $[0,1]$ such that f_n converges uniformly to f. Show that $f \in C([0,1])$.

(b) Assume without proof that

$$\|f\|_\infty := \sup\{|f(t)| : t \in [0,1]\}$$

is a norm on $C([0,1])$. Show that $C([0,1])$ is a Banach space with respect to $\|\cdot\|_\infty$.

Notes and Comments

Proof of (a). To show that $f \in C([0,1])$, we must show that, for any $p \in [0,1]$ and any $\varepsilon > 0$, $\exists \delta > 0$ such that, whenever $|x - p| < \delta$, we have $|f(x) - f(p)| < \varepsilon$.

Let $p \in [0,1]$ and $\varepsilon > 0$. Since $\{f_n\} \overset{\text{unif.}}{\to} f$, $\exists N \in \mathbb{Z}_+$ such that for all $m > 0$ and all $x \in [0,1]$,

$$\boxed{|f_n(x) - f(x)| < \frac{\varepsilon}{3}} (*).$$

Let $M > N$. Then, by $(*)$,

$$|f_M(x) - f(x)| < \frac{\varepsilon}{3} \text{ and } |f_M(p) - f(p)| < \frac{\varepsilon}{3}.$$

Since f_M is continuous, $\exists \delta$ such that

$$|x - p| < \delta \Longrightarrow |f_M(x) - f_M(p)| < \frac{\varepsilon}{3}.$$

Thus, whenever $|x - p| < \delta$,

$$|f(x) - f_M(x)| + |f_M(x) - f_M(p)| + |f_M(p) - f(p)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So, by the triangle inequality, we have

$$|f(x) - f(p)| \leq |f(x) - f_M(x)| + |f_M(x) - f_M(p)| + |f_M(p) - f(p)| < \varepsilon.$$

Hence f is continuous at p. Since p was arbitrary, f is continuous on $[0,1]$, i.e., $f \in C([0,1])$. □

Proof of (b). To show that $C([0,1])$ is a Banach space with respect to $\|\cdot\|_\infty$, it suffices to show that every Cauchy sequence converges to an element in $C([0,1])$ with respect to $\|\cdot\|_\infty$.

Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence with respect to $\|\cdot\|_\infty$. Then, for any $\varepsilon > 0$, $\exists N \in \mathbb{Z}_+$ such that, for all $n, m > N$,

$$\|f_n - f_m\|_\infty < \varepsilon.$$

Thus for any $p \in [0,1]$, the sequence $\{f_n(p)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ (a complete metric space) and so $\{f_n(p)\}_{n=1}^\infty$ converges to some value $f(p)$. Thus $\{f_n\}_{n=1}^\infty$ converges pointwise to some function f on $[0,1]$.

Claim: The sequence $\{f_n\}_{n=1}^\infty$ converges uniformly to f.

Proof of Claim. Let $\varepsilon > 0$. Then $\exists N \in \mathbb{Z}_+$ such that, for all $n, m > N$,

$$\boxed{\|f_n - f_m\|_\infty < \varepsilon} (*).$$

For $M > N$ and $p \in [0,1]$, $|f_M(p) - f_n(p)| < \varepsilon$ for $n > N$ by $(*)$. Thus

$$\{f_n(p) \mid n > N, p \in [0,1]\} \subseteq (f_M(p) - \varepsilon, f_M(p) + \varepsilon)$$

and so $f(p) \in [f_M(p) - \varepsilon, f_M(p) + \varepsilon]$. Hence, for any $n > M$, $f_n(p) \in [f(p) - \varepsilon, f(p) + \varepsilon]$ for all p. That is, $\boxed{|f_n(p) - f(p)| < \varepsilon}$ for all p. Since ε was arbitrary, $\{f_n\}_{n=1}^\infty$ converges uniformly to f. □

By the claim and part (a), f is continuous. Thus every Cauchy sequence converges and so $C([0,1])$ is a Banach space with respect to $\|\cdot\|_\infty$. □

Problem 5.1.4

Let $\mathcal{H}$ be a complex Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a linear map.

(a) Show that if T is bounded, then there is a linear map $S : \mathcal{H} \to \mathcal{H}$ such that $(Tv \mid w) = (v \mid Sw)$ for all $v, w \in \mathcal{H}$. (In other words, show that T has an adjoint.)

(b) Conversely, show that if there is a (not necessarily bounded) map $S : \mathcal{H} \to \mathcal{H}$ such that $(Tv \mid w) = (v \mid Sw)$ for all $v, w \in \mathcal{H}$, then T is bounded.

Notes and Comments

Proof of (a). Let $y \in \mathcal{H}$ and define $S_y : \mathcal{H} \to \mathbb{C}$ by $S_y(x) = (Tx \mid y) \in \mathbb{C}$. Then, since T is linear, S_y is also linear. Furthermore,

$$|S_y(x)| = |(T(x) \mid y)| \overset{CS\leq}{\leq} \|T(x)\|\|y\| \overset{T \text{ bdd}}{\leq} \|T\|\|x\|\|y\| = \underbrace{\|T\|\|y\|}_{M}\|x\|.$$

Hence S_y is bounded. Thus $S_y \in \mathcal{H}^*$, the dual of $\mathcal{H}$. By the Riesz–Fréchet Theorem, there is a unique $z_y \in \mathcal{H}$ such that

$$(T(x) \mid y) = S_y(x) = (x \mid z_y)$$

for all $x \in \mathcal{H}$. Define $S : \mathcal{H} \to \mathcal{H}$ by $\boxed{S(y) = z_y}$. Then S has the adjoint property.

We now wish to show that S is a linear map. Indeed, for $y_1, y_2 \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\begin{aligned}(x \mid S(\alpha y_1 + \beta y_2)) &\overset{unique}{=} (T(x) \mid \alpha y_1 + \beta y_2)\\ &= \overline{\alpha}(T(x) \mid y_1) + \overline{\beta}(T(x) \mid y_2)\\ &\overset{unique}{=} \overline{\alpha}(x \mid S(y_1)) + \overline{\beta}(x \mid S(y_2))\\ &= (x \mid \alpha S(y_1) + \beta S(y_2))\end{aligned}$$

for all $x \in \mathcal{H}$. Using conjugate linearity in the second component, we find that $S(\alpha y_1 + \beta y_2) - (\alpha S(y_1) + \beta S(y_2))$ is orthogonal to $\mathcal{H}$ for all $x \in \mathcal{H}$. That is, this quantity must be 0 and so S is linear. $\square$

Proof of (b). This result is a quick application of the Closed Graph Theorem. Suppose $x_n \to x$ and $T(x_n) \to y$. We want to show that $T(x) = y$. For any $z \in \mathcal{H}$, we have

$$(T(x) \mid z) = (x \mid S(z)) = \lim_{n\to\infty}(x_n \mid S(z)) = \lim_{n\to\infty}(T(x_n) \mid z) = (y \mid z).$$

Thus $T(x) - y$ is orthogonal to all of $\mathcal{H}$. That is, $T(x) = y$ and so the graph of T is closed. By the Closed Graph Theorem, as $\mathcal{H}$ is (more than) a Banach space, T is continuous (hence bounded). $\square$

Problem 5.1.5

Let f be a complex function on an open connected subset Ω of the complex plane.

(a) What are the Cauchy–Riemann equations for f at $z_0 \in \Omega$?

(b) Discuss the existence of the complex derivative $f'(z_0)$ in terms of the Cauchy–Riemann equations at z_0. (Ideally, you should give both necessary as well as sufficient conditions for $f'(z_0)$ to exist. Note that you are not asked to prove anything here.)

(c) Show that a real-valued function on Ω is holomorphic if and only if it is constant.

Notes and Comments

Proof of (a). Write $f = u + iv$ where u and v are real-valued. Then the Cauchy–Riemann equations are $u_x = v_y$ and $-u_y = v_x$. □

Proof of (b). As above, write $f = u + iv$.

Necessary condition: If f is differentiable at z_0 then $f_{\mathbb{R}}$ (f viewed as a map $\mathbb{R}^2 \to \mathbb{R}$) is differentiable at z_0 and u, v satisfy the Cauchy–Riemann equations.

Sufficient condition: If u, v (viewed as maps $\mathbb{R}^2 \to \mathbb{R}$) have 1st-order partials in a neighborhood of $z_0 = (x_0, y_0)$, are continuous at z_0, and satisfy the Cauchy–Riemann equations, then f is differentiable at z_0. □

Proof of (c). If f is constant then it is obviously differentiable (with derivative 0).

Conversely, if it is holomorphic on Ω, then $f = u + iv$ satisfies the Cauchy–Riemann equations. Since f is real-valued, $v \equiv 0$. Hence the Cauchy-Riemann equations guarantee $u_x = 0 = u_y$ since $v = 0$. Thus u is a constant function. □

Problem 5.1.6

Let H be a complex Hilbert space and $T : H \to H$ a linear map.

(a) What does it mean for T to be bounded?

(b) Define the operator norm, $||T||$, of T and show that $\|Th\| \leq \|T\| \cdot \|h\|$ for all $h \in H$.

(c) Show that T is bounded if and only if T is continuous from $\mathcal{H}$ to $\mathcal{H}$.

Notes and Comments

Solution to (a). T is *bounded* if there exists $M > 0$ such that $||Tx|| \leq M||x||$ for all $x \in H$. □

Solution to (b). The *operator norm* is $||T|| = \sup_{x \in H}\{||Tx|| : ||x|| = 1\}$. By submultiplicativity of the operator norm,

$$||T\frac{x}{||x||}|| \leq ||T||.$$

Multiplying through by $||x||$ gives the desired result. □

Proof of (c). Assume that T is bounded and let $\varepsilon > 0$ be arbitrary. For any x and y with $||x - y|| < \frac{\varepsilon}{||T||}$, we have

$$||Tx - Ty|| = ||T(x - y)|| \leq ||T||||x - y|| = ||T||\frac{\varepsilon}{||T||} = \varepsilon$$

and so T is actually Lipschitz continuous (hence continuous).

Conversely, assume that T is continuous. Then, in particular, it is continuous at 0. Let $\varepsilon = 1$. Then $\exists\delta$ such that, for all x with $||x|| < \delta$, we have $||Tx|| < 1$. By scaling x with $\frac{1}{\delta}$ we can extend this to the unit ball, thus completing the proof. □

Analysis Exam

Fall 2012

Problem 5.2.1

State the Hahn–Banach Theorem and use it to show that if B is a Banach space, then its dual, B^**, of bounded linear functionals separates points of B. (That is, you are asked to show that if a and b are distinct elements of B, then there is a $\phi \in B^*$ such that $\phi(a) \neq \phi(b)$.)

Notes and Comments

Theorem 5.2.1 (Hahn–Banach): *Let V be a Banach space and $W \subseteq V$ a subspace of V. Then for any $\varphi \in W^*$ there exists a $\overline{\varphi} \in V^*$ such that $\overline{\varphi}|_W = \varphi$ and $||\overline{\varphi}|| = ||\varphi||$.*

Proof. To separate the points of B, we let $a \neq b \in B$ be arbitrary and exhibit $\varphi \in B^*$ such that $\varphi(a) \neq \varphi(b)$. Define $\hat{\varphi}$ on span $\{a - b\}$ by $\hat{\varphi}(a - b) = ||a - b|| \neq 0$. By Hahn–Banach, we can extend $\hat{\varphi}$ to a functional φ on all of B with the property that $\varphi(a - b) = \varphi(a) - \varphi(b) = ||a - b|| \neq 0$ as desired. □

Problem 5.2.2

State the Residue Theorem (from Complex Analysis) and use it to evaluate $\displaystyle\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} dx$ for $a > 0$. Be sure to justify any limits required.

Notes and Comments

Theorem 5.2.2 (Residue Theorem): *Let D be a domain and suppose $f : D \to \mathbb{C}$ is analytic except for a finite collection of isolated singularities at $z_1, \ldots, z_n \in D$. Let γ be a closed curve in D that is homotopically trivial in D and assume $z_j \notin \gamma$ for all j. Then*

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_{j=1}^n \operatorname{ind}(\gamma; z_j) \operatorname{Res}(f; z_j)$$

Proof. Consider the function $f(z) = \dfrac{x^2}{(x^2 + a^2)^2}$ which has poles at $\pm ia$ and is holomorphic on the rest of $\mathbb{C}$. As f is even, we can integrate over all of $\mathbb{R}$ and divide the result by two. Define the following paths:

- $\alpha_R(t) = Re^{\pi i t}$ $(t \in [0, 1])$ is the semi-circle of radius R centered at the origin in the upper half plane, oriented counterclockwise,
- $\beta_R(t) = 2Rt - R$ $(t \in [0, 1])$ is the line from $-R$ to R,
- γ_R is the concatenation of α_R and β_R, oriented counterclockwise. For $R > |a|$, we have

$$\int_{\beta_R} f(z)\, dz = \int_{\gamma_R} f(z)\, dz - \int_{\alpha_R} f(z)\, dz.$$

Note that $\boxed{\int_{\alpha_R} f(z)\ dz \to 0}(*)$ as $R \to \infty$. That is, for sufficiently large R, $|f(\gamma_R(t))| < \frac{1}{|\gamma_R(t)|^2}$. Hence $\left|\int_{\alpha_R} f(z)\ dz\right| < \int_{\alpha_R} \left|\frac{1}{z^2}\right|\ dz$ and this does indeed go to 0.

Now we need only evaluate the first integral. To do so, we use the Residue Theorem. The index is the winding number, so $\boxed{\operatorname{ind}(\gamma_R, ia) = 1}$. Since ai is the only pole and it has order two, we have

$$\begin{aligned}
\operatorname{Res}(f, ai) = \frac{1}{(2-1)!} \lim_{z\to ai} \frac{d}{dz}\left((z-ai)^2 \frac{z^2}{(z^2+a^2)^2}\right) &= \lim_{z\to ai} \frac{d}{dz} \frac{z^2}{(z+ai)^2} \\
&= \lim_{z\to ai} \frac{2z(z+ai)^2 - 2(z+ai)z^2}{(z+ai)^4} \\
&= \frac{-i}{4a}
\end{aligned}$$

Finally, we can compute the integral

$$\begin{aligned}
\int_0^\infty f(x)dx = \frac{1}{2}\int_{\mathbb{R}} f(z)dz = \frac{1}{2}\lim_{R\to\infty}\left(\int_{\gamma_R} f(z)dz - \int_{\alpha_R} f(z)dz\right) &\overset{(*)}{=} \frac{1}{2}\lim_{R\to\infty}\int_{\gamma_R} f(z)dz \\
&= \frac{1}{2}\lim_{R\to\infty} 2\pi i \operatorname{ind}(\gamma_R; ai) \operatorname{Res}(f; ai) \\
&= \pi i \lim_{R\to\infty} \frac{-i}{4a} \\
&= \frac{\pi}{4a}
\end{aligned}$$

Thus $\boxed{\int_0^\infty f(x)\ dx = \frac{\pi}{4a}}$. □

Problem 5.2.3

Consider a power series $\sum_{n=1}^\infty a_n x^n (\dagger)$ **for real constants** $a_n \in \mathbb{R}$**. Show that there is a** $\rho \in [0, \infty]$ **such that either**

(i) $\rho = 0$ **by which we mean** $(\dagger)$ **converges only for** $x = 0$**, or**

(ii) $\rho = \infty$ **by which we mean** $(\dagger)$ **converges absolutely for all** x**, or**

(iii) $0 < \rho < \infty$ **and** $(\dagger)$ **converges absolutely if** $|x| < \rho$ **and diverges if** $|x| > \rho$**.**

Give examples (with all $a_n \neq 0$**) where** $\rho = 0, \rho = \infty$**, and** $0 < \rho < \infty$**.**

Notes and Comments

Proof. Define $\boxed{\rho = \sup\{r \in \mathbb{R} \ :\ (a_n r^n)_{n=1}^\infty \text{ is bounded}\}}$.

If $\rho = 0$, then $\sum_{n=1}^\infty a_n x^n$ diverges for all $x \neq 0$ since the sequence $(a_n x^n)_{n=1}^\infty$ is unbounded. If $\rho = \infty$ or $0 < \rho < \infty$, choose any $r < \rho$. We will show that the series converges absolutely on $\overline{D_r(0)}$.

Let R be such that $r < R < \rho$. By definition of ρ, we have that $(a_n R^n)_{n=1}^{\infty}$ is bounded and hence there exists $M \in \mathbb{N}$ such that $|a_n|R^n \leq M$. For any $x \in \overline{D_r(0)}$, we have

$$|a_n x^n| \leq |a_n| r^n = |a_n| R^n \left(\frac{r^n}{R^n}\right) \leq M \left(\frac{r}{R}\right)^n.$$

As $R > r$, we have $\frac{r}{R} < 1$. Hence

$$\sum_{n=1}^{\infty} |a_n x^n| \leq \sum_{n=1}^{\infty} M \left(\frac{r}{R}\right)^n = \frac{M}{1-(r/R)}.$$

Thus the series converges absolutely on $\overline{D_r(0)}$ as desired.

For the converse (for (iii)), note that $|x| = \rho$ will give us a geometric series ($r = 1$). That is, the series will blow up.

Examples

(i) $a_n = 2^{n!}$

(ii) $a_n = 2^{-n!}$

(iii) $a_n = (-1)^n$ (here $\rho = 1$)

□

Problem 5.2.4

Give a precise statement of the theorem which implies that a holomorphic function on an open subset of the complex plane is locally represented by a power series. Use your theorem to calculate the radius of converges of the MacLaurin series for $f(z) = \frac{1}{1+e^z}$. (The MacLaurin series is just the Taylor series for f about $z = 0$.)

Notes and Comments

Theorem 5.2.3 *Let U be an open subset of $\mathbb{C}$ and $f \in \mathcal{H}(U)$. Then, for each $z_0 \in U$, there exists a sequence (a_n) and a radius $r > 0$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for $|z - z_0| < r$.*

Proof. Note that f has poles at $z = \pm\pi i$. Hence, about $z_0 = 0$, f has a radius of convergence $r = \pi$. □

Problem 5.2.5

Let μ be a measure on the Borel sets of $\mathbb{R}$ such that, for any Borel set $E \subseteq \mathbb{R}$, we have

$$\mu(E) = \inf\{\mu(U) \ : \ U \text{ is an open set containing } E\}$$

and $\mu([a,b]) < \infty$ for an interval $[a,b]$.

(i) Show that for any $\varepsilon > 0$ there is an open set O and a closed set C such that $C \subseteq E \subseteq O$ and $\mu(O \setminus C) < \varepsilon$.

(ii) Using the above, show that there are Borel sets G and F such that $F \subseteq E \subseteq G$ with $\mu(G \backslash F) = 0$.

(Hint: Finding a neighborhood O of E such that $\mu(O \setminus E) < \varepsilon$ is pretty easy if $\mu(E) < \infty$.)

Notes and Comments

Proof of (i). Note that μ is σ-finite because closed intervals have finite measures.

Claim: For any Borel set E, we can find an open set U with $E \subseteq U$ such that $\mu(U \setminus E) < \varepsilon$.

Proof. Let $\varepsilon > 0$.[1]

If $\mu(E) < \infty$ then, by definition of μ, there is an open set U with $\mu(U) < \mu(E) + \varepsilon$. Since measures are additive on disjoint sets, we have

$$\mu(U \cap E) + \mu(U \setminus E) = \mu(U) < \mu(E) + \varepsilon.$$

As $E \subseteq U$, we get

$$\mu(E) + \mu(U \setminus E) < \mu(E) + \varepsilon.$$

Hence $\boxed{\mu(U \setminus E) < \varepsilon}$.

Assume $\mu(E) = \infty$. By σ-finiteness, $E = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$ for all n. For each E_n, choose U_n such that $\mu(U_n \setminus E_n) < \frac{\varepsilon}{2^n}$. Let $U = \bigcup_{n=1}^{\infty} U_n$. Then, by subadditivity of μ and since $U_n \setminus E \subseteq U_n \setminus E_n$, we have

$$\mu(U \setminus E) \le \sum_{n=1}^{\infty} \mu(U_n \setminus E) \le \sum_{n=1}^{\infty} \mu(U_n \setminus E_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

That is, $\boxed{\mu(U \setminus E) < \varepsilon}$. □

Now we can tackle the general problem. By the claim, there is an open set O such that $\boxed{\mu(O \setminus E) < \frac{\varepsilon}{2}}$. Consider E^c. Since the Borel sets form a σ-algebra, E^c is also a Borel set. Hence, by the claim, we have an open set U such that $\boxed{\mu(U \setminus E^c) < \frac{\varepsilon}{2}}$.

Let $C = U^c$. Then C is a closed set and

$$\mu(E \setminus C) = \mu(E \cap C^c) = \mu(E \cap U) = \mu(U \setminus E^c) < \frac{\varepsilon}{2}.$$

Since $U \setminus C = ((U \cap E) \setminus C) \cup ((U \setminus E) \setminus C) = (E \setminus C) \cup (U \setminus E)$, subadditivity tells us that

$$\mu(U \setminus C) \le \mu(E \setminus C) + \mu(U \setminus E) < \varepsilon$$

as desired. □

[1] How else would a good analysis proof start?

Proof of (ii). By part (i), there are open sets O_n and closed sets C_n with $C_n \subseteq E \subseteq O_n$ such that $\boxed{\mu(O_n \setminus C_n) < \frac{1}{n}}$. Define $F = \bigcup_{n=1}^{\infty} C_n$ and $G = \bigcap_{n=1}^{\infty} O_n$. Since Borel sets form a σ-algebra, F and G are both Borel sets. Note that $G \setminus F \subseteq O_n \setminus C_n$ for all n.[2] Thus

$$\mu(G \setminus F) \leq \mu(O_n \setminus C_n) < \frac{1}{n}$$

for all n. Hence $\mu(G \setminus F) = 0$ as desired. □

Problem 5.2.6

Show that a continuous function $f : (0, 1] \to \mathbb{R}$ is uniformly continuous if and only if there is a continuous extension $g : [0, 1] \to \mathbb{R}$. (That is, g is a continuous function such that $g(x) = f(x)$ for all $x \in (0, 1]$.)

Notes and Comments

Proof. ($\Rightarrow$): If g is a continuous extension of f to $[0, 1]$, then g is uniformly continuous by the Heine–Cantor Theorem (i.e., since $[0, 1]$ is compact). Since $f = g|_{(0,1]}$, f is also uniformly continuous.

($\Leftarrow$): Consider a Cauchy sequence $\{x_n\} \subseteq (0, 1]$ such that $x_n \to 0$. Since f is uniformly continuous, it maps Cauchy sequences to Cauchy sequences. Thus $\{f(x_n)\}$ is a Cauchy sequence.

By the completeness of $\mathbb{R}$, $f(x_n) \to a \in \mathbb{R}$. Define $g(0) = a$. From the uniqueness of limits, g is continuous at 0; hence on $[0, 1]$. Thus g is the desired continuous extension of f. □

[2] It can take a minute to realize this, depending on when you're thinking about it.

Analysis Exam

Summer 2013

Problem 5.3.1

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is everywhere analytic (i.e., entire).

(a) Show that the function $g(z) = f(\overline{z})$ is entire only if f is a constant function.

(b) Show that the function $h(z) = \overline{f(\overline{z})}$ is entire.

Notes and Comments

Proof of (a). Write $f = u + iv$ where u, v are real-valued functions. Since f is entire, the Cauchy–Riemann equations give

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0)$$

for all $(x_0, y_0) \in \mathbb{R}^2$.

Write $g(x, y) = f(x, -y) = u(x, -y) + iv(x, -y)$ and assume g is entire. Then the Cauchy–Riemann equations hold and so

$$u_x(x_0, -y_0) = -v_y(x_0, -y_0) \text{ and } -u_y(x_0, -y_0) = -v_x(x_0, y_0).$$

However, by the Cauchy–Riemann equations for f, we have

$$u_x(x_0, -y_0) = -v_y(x_0, -y_0) = -u_x(x_0, -y_0) \text{ and } -u_y(x_0, -y_0) = -v_x(x_0, y_0) = u_y(x_0, -y_0).$$

Hence $u_x = u_y = 0$ on all of $\mathbb{R}^2$. Thus u is constant on $\mathbb{R}^2$. Similarly v is constant on $\mathbb{R}^2$. Hence f is constant as desired. □

Proof of (b). Write f as in part (a). Then $h(x, y) = u(x, -y) - iv(x, -y)$. Checking the partials of the real and imaginary parts of h, the Cauchy–Riemann equations hold. Also, the partials are continuous since u, v are smooth (and hence have continuous partials). Thus h is differentiable at any point in $\mathbb{C}$, i.e., h is entire. □

Problem 5.3.2

Let $C[0, 1]$ denote the vector space of all continuous complex-valued functions $f : [0, 1] \to \mathbb{C}$.

Show that

$$S = \{f \in C[0, 1] : f(0) = 0\}$$

is a linear subspace of $C[0, 1]$. Give $C[0, 1]$ the supremum (uniform) norm $\|\cdot\|_\infty$:

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.$$

Is S a closed subspace? Why or why not?

Notes and Comments

Proof. Since $C[0,1]$ is a vector space, we know that the scalar product and sum of continuous functions are continuous. For any functions $f, g \in C[0,1]$ such that $f(0) = g(0) = 0$, we have

$$af(0) + bg(0) = a0 + b0 = 0.$$

So S is a linear subspace of $C[0,1]$.

S is a closed subspace under $\|\cdot\|_\infty$: Assume $(f_n)_{n=1}^\infty$ is a sequence in S that converges to $f \in C[0,1]$. We want to show that $f \in S$. By definition, $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. For any $x \in [0,1]$, we have $|f(x) - f_n(x)| \le \|f - f_n\|_\infty$. As $f_n \in S$, we have $f_n(0) = 0$. Thus, for all n,

$$|f(0)| = |f(0) - f_n(0)| \le \|f - f_n\|_\infty .$$

Taking limits, $|f(0)| \le 0$. Hence $f(0) = 0$ and we have $f \in S$. Thus S contains all its limit points and is thus closed. ☐

Problem 5.3.3

Let (X, M, μ) be a measure space. Let $h : X \to [0, \infty]$ be an M-measurable function on X. Define $\lambda : M \to [0, \infty]$ by

$$\lambda(E) = \int_E h \, d\mu.$$

Show that λ is a measure on (X, M).

Notes and Comments

Proof. Given that $h \ge 0$, we know that $\lambda(E) = \int_E h \, d\mu \ge 0$. Note that

$$\lambda(\varnothing) = \int_\varnothing h \, d\mu = \int_X h \cdot \chi_\varnothing = 0.$$

We now prove that λ is countably additive on disjoint sets. Let $\{E_j\}_{j=1}^\infty$ be disjoint sets in M. Define $h_j = h \cdot \chi_{E_j}$ for $j \in \mathbb{N}$, let $f_n = \sum_{j=1}^n h_j$ for $n \in \mathbb{N}$, and let $f = \lim\limits_{n\to\infty} f_n$.

Since $f_n \le f_{n+1}$ for each n, the Monotone Convergence Theorem implies that

$$\int_X f \, d\mu \overset{(MCT)}{=} \lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \int_X \sum_{j=1}^n h_j \, d\mu = \lim_{n\to\infty} \int_X \sum_{j=1}^n h \cdot \chi_{E_j} \, d\mu.$$

Therefore

$$\int_X f \, d\mu = \lim_{n\to\infty} \left(\sum_{j=1}^n \int_X h \cdot \chi_{E_j} \, d\mu \right) = \lim_{n\to\infty} \left(\sum_{j=1}^n \lambda(E_j) \right) = \sum_{j=1}^\infty \lambda(E_j).$$

Thus $\boxed{\int_X f \, d\mu = \sum_{j=1}^\infty \lambda(E_j)}$ $(*)$. On the other hand, $\sum_{j=1}^\infty h \cdot \chi_{E_j} \equiv 0$ on $X \setminus \bigcup_{j=1}^\infty E_j$. Since $\{E_j\}_{j=1}^\infty$ is a disjoint collection, we have $\sum_{j=1}^\infty h \cdot \chi_{E_j} \equiv h$ on $\bigcup_{j=1}^\infty E_j$. Hence

$$\sum_{j=1}^\infty \lambda(E_j) \overset{(*)}{=} \int_X f \, d\mu = \int_X \left(\sum_{j=1}^\infty h \cdot \chi_{E_j} \right) d\mu = \int_{\bigcup_{j=1}^\infty E_j} h \, d\mu = \lambda\left(\bigcup_{j=1}^\infty E_j \right).$$

Thus λ is a measure on (X, M). ☐

Problem 5.3.4

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$. If S is any nonempty subset of $\mathcal{H}$ and V the closed subspace generated by S, i.e., $V = \overline{\text{Span}(S)}$, show that $S^\perp = V^\perp$, i.e., their orthogonal complements are equal.

Notes and Comments

Proof. We show that $S^\perp = V^\perp$ by showing both set containments.

$(S^\perp \supseteq V^\perp)$: Let $v^\perp \in V^\perp$. Then for any $v \in V$, we have $(v, v^\perp) = 0$ by definition. Since $S \subset V$, we have $(s, v^\perp) = 0$ for all $s \in S$. Thus $v^\perp \in S^\perp$. That is, $\boxed{S^\perp \supseteq V^\perp}$.

$(S^\perp \subseteq V^\perp)$: Let $s^\perp \in S^\perp$. Then $(s, s^\perp) = 0$ for all $s \in S$.

Consider $x \in \text{Span}(S)$. Then $x = \sum_{i=1}^n \lambda_i s_i$ for $s_i \in S$. By linearity[3] of $(\cdot,\cdot)$,

$$(x, s^\perp) = \left(\sum_{i=1}^n \lambda_i s_i, s^\perp\right) = \sum_{i=1}^n \lambda_i (s_i, s^\perp) = \sum_{i=1}^n \lambda_i \cdot 0 = 0.$$

Thus $\boxed{(x, s^\perp) = 0}(*)$ for all $x \in \text{Span}(S)$.

Let $v \in V$. Then there is a sequence $\{v_n\}_{n=1}^\infty$ in $\text{Span}(S)$ such that $v_n \to v$. By continuity of the inner product,

$$(v, s^\perp) = \left(\lim_{n\to\infty} v_n, s^\perp\right) = \lim_{n\to\infty} (v_n, s^\perp) \overset{(*)}{=} \lim_{n\to\infty} 0 = 0.$$

Thus $(v, s^\perp) = 0$. That is, $\boxed{S^\perp \subseteq V^\perp}$. □

Problem 5.3.5

Let $\{a_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}$. We state two definitions of $\limsup a_n$ below. Show definition (a) implies the statement in (b). (You don't have to prove the converse.)

(a) $\limsup a_n = \lim_{n\to\infty} (\sup\{a_k : k \geq n\})$.

(b) $\limsup a_n$ **is the largest subsequential limit of** $\{a_n\}_{n=1}^\infty$**. (Recall that** $a \in [-\infty, \infty]$ **is said to be a subsequential limit of** $\{a_n\}_{n=1}^\infty$ **if some subsequence** $\{a_{n_k}\}_{k=1}^\infty$ **satisfies** $\lim_{k\to\infty} a_{n_k} = a$**.)**

Notes and Comments

Proof. Let $b_n = \sup\{a_k : k \geq n\}$. Observe that (a) implies that $\boxed{b_n \searrow \limsup a_n}$. We proceed by considering the different possibilities for $\limsup a_n$.

(I) $\boxed{\limsup a_n = -\infty}$.

For n fixed, $\inf_{k\geq n}\{a_k\} \leq \sup_{k\geq n}\{a_k\}$, so

$$-\infty \leq \liminf a_n = \lim_{n\to\infty}\left(\inf_{k\geq n}\{a_k\}\right) \leq \lim_{n\to\infty}\left(\sup_{k\geq n}\{a_k\}\right) = \limsup a_n = -\infty.$$

This means that equality holds throughout, so $-\infty$ is the only subsequential limit.

[3] In at least the first component.

(II) $\boxed{\limsup a_n = \infty}$.

Given that $b_n \searrow \limsup a_n = \infty$, it must be that $b_n = \infty$ for all n. Therefore it is possible to construct $a_{n_k} \to \infty$ as $k \to \infty$ (at each stage, choose a term larger than the previous). No subsequential limit is larger than ∞.[4]

(III) $\boxed{\limsup a_n = a \in \mathbb{R}}$.

Claim: No subsequential limit is larger than a.

Proof. Suppose $b > a$ is a subsequential limit. Since $b_n \searrow \limsup a_n = a$, there exists n such that $a \leq b_n < b$, so $a \leq \sup\{a_k : k \geq n\} < b$. This means that $a_k \leq b_n < b$ for $k \geq n$, so $b_n < a_k$ holds only for finitely many values of k. Hence, no subsequence of $\{a_k\}$ converges to b. □

Claim: a is a subsequential limit.

Proof. The idea is to inductively construct subsequences $\{a_{n_l}\}_{l=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ and $\{b_{k_l}\}_{l=1}^{\infty}$ of $\{b_n\}_{n=1}^{\infty}$ that are "very close," and then take advantage of the fact that $b_n \to a$.

Since $b_n \searrow a$, we may choose k_1 such that $a \leq b_{k_1} < a+1$. Now, given that $b_{k_1} = \sup\{a_n : n \geq k_1\}$, we may choose $n_1 \geq k_1$ such that $|b_{k_1} - a_{n_1}| < 1$.

(Inductive hypothesis) Assume there are

$$k_1 \leq n_1 < k_2 \leq n_2 < \cdots < k_l \leq n_l$$

such that $\boxed{a \leq b_{k_j} < a + \frac{1}{j}}$ and $\boxed{\left|b_{k_j} - a_{n_j}\right| < \frac{1}{j}}$.

There exists $b_{k_{l+1}}$ such that $a \leq b_{k_{l+1}} < a + \frac{1}{l+1}$. In fact, since $b_n \searrow a$, we may choose k_{l+1} so that $k_{l+1} > n_l$. Now, choose $a_{n_{l+1}}$ such that $n_{l+1} \geq k_{l+1}$ and $\left|b_{k_{l+1}} - a_{n_{l+1}}\right| < \frac{1}{l+1}$. Again, it is possible to choose n_{l+1} because $b_{k_{l+1}} = \sup\{a_n : n \geq k_{l+1}\}$.

Thus, we have inductively constructed subsequences $\{a_{n_l}\}_{l=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$and $\{b_{k_l}\}_{l=1}^{\infty}$ of $\{b_n\}_{n=1}^{\infty}$ such that $|a - b_{k_l}| < \frac{1}{l}$ and $|b_{k_l} - a_{n_l}| < \frac{1}{l}$ for all l.

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\frac{2}{N} < \varepsilon$. For $l \geq N$,

$$|a_{n_l} - a| \leq |a_{n_l} - b_{k_l}| + |b_{k_l} - a| < \frac{2}{l} \leq \frac{2}{N} < \varepsilon.$$

We conclude that $a_{n_l} \to a$ as $l \to \infty$. □

Thus $\limsup a_n$ is the largest subsequential limit in all cases. □

[4]For obvious reasons.

Problem 5.3.6

Let V and W be Banach spaces. A bounded linear operator $A \in L(V,W)$ is said to be *bounded below* **if there is a constant $C > 0$ such that**

$$\|A(x)\|_W \geq C \|x\|_V\,, \qquad \forall x \in V.$$

(a) Show that if A is bounded below, then A is injective and has closed range.

(b) Show that if A is bounded below then $A^{-1} : \operatorname{Range}(A) \to V$ is bounded. Thus, if A has dense range then $A^{-1} \in L(W,V)$.

Notes and Comments

Proof of (a). First we show that A is injective. Since A is linear, it suffices to show that $\ker A = \{0\}$. If $A(x) = 0$ then, by assumption, $0 = \|A(x)\|_W \geq C\|x\|_V$. That is, $\|x\|_V = 0$ and so $x = 0$. Thus $\ker A = \{0\}$ as desired.

To show that A has closed range, consider a sequence $(y_i)_{i=1}^\infty$ in $\operatorname{Range}(A)$ and suppose it converges to y. Then $(y_i)_{i=1}^\infty$ is Cauchy. Since A is injective, $\exists! x_i \in V$ such that $A(x_i) = y_i$ for all $i \in \mathbb{N}$.

We claim that $(x_i)_{i=1}^\infty$ is Cauchy. Since A is bounded below by C, observe that

$$\|y_n - y_m\|_W = \|A(x_n - x_m)\|_W \geq C\|x_n - x_m\|_V\,.$$

Let $\varepsilon > 0$. Since $(y_i)_{i=1}^\infty$ is Cauchy, $\exists N$ such that, for all $n, m \geq N$, we have $\|y_n - y_m\|_W < C\varepsilon$. Thus, by the above,

$$C\|x_n - x_m\|_V < C\varepsilon \Rightarrow \boxed{\|x_n - x_m\|_V < \varepsilon}.$$

Thus $(x_i)_{i=1}^\infty$ is Cauchy. Since V is a Banach space, $x_i \to x \in V$. Since A is bounded, it is a continuous map. Hence

$$y = \lim_{i\to\infty} y_i = \lim_{i\to\infty} A(x_i) = A\left(\lim_{i\to\infty} x_i\right) = A(x).$$

Thus $y \in \operatorname{Range}(A)$ as desired. □

Proof of (b). Let $y \in \operatorname{Range}(A)$. By part (a), $A^{-1}(y)$ is well-defined. Since A is bounded below,

$$\|y\|_W \geq C\left\|A^{-1}(y)\right\|_V\,.$$

Thus, since $C > 0$, we have $\boxed{\frac{1}{C}\|y\|_W \geq \left\|A^{-1}(y)\right\|_V}$. That is, $\|A^{-1}\| \leq \frac{1}{C}$ and so $A^{-1} : \operatorname{Range}(A) \to V$ is bounded.

Now further assume that A has dense range. By part (a), A has closed range. Thus $\operatorname{Range}(A) = \overline{\operatorname{Range}(A)} = W$. Hence $A^{-1} \in L(W,V)$ as desired. □

Analysis Exam

Fall 2013

Problem 5.4.1

Suppose f is entire and $\lim_{z\to\infty} f(z) \in \mathbb{C}$ exists. Show that f is constant.

Notes and Comments

Proof. Assume $\lim_{z\to\infty} f(z) = z_0 \in \mathbb{C}$. Then $\exists R > 0$ such that, for z with $|z| > R$, we have $|f(z) - z_0| < 42$. That is, outside of the disk of radius R about 0, f is bounded by $42 + |z_0|$. On the closed disk $\overline{D_R(0)}$, f obtains a maximum M. That is, $f(z) \leq \max M, 42 + |z_0|$ for all $z \in \mathbb{C}$. Hence f is a bounded entire function. By Liouville's Theorem, f is constant. □

Problem 5.4.2

Let $(V, (\cdot,\cdot))$ be an inner product space over the field $\mathbb{F}$.

(a) If $\mathbb{F} = \mathbb{R}$, show that vectors $x, y \in V$ are orthogonal *if and only if*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 .$$

(b) Show that (a) is *false* **for any complex ($\mathbb{F} = \mathbb{C}$) inner product space V, where x can be** *any* **nonzero vector in V. (Hint: y should be more imaginary than x.)**

Notes and Comments

Proof of (a). ($\Rightarrow$) Assume that x, y are orthogonal, so $\boxed{(x,y) = 0}(*)$. Then

$$\begin{aligned}
\|x + y\|^2 &= (x + y, x + y) \\
&= (x, x + y) + (y, x + y) \\
&= (x, x) + (x, y) + (y, x) + (y, y) \\
&= \|x\|^2 + (x, y) + (y, x) + \|y\|^2 \\
&\overset{(*)}{=} \|x\|^2 + \|y\|^2 .
\end{aligned}$$

($\Leftarrow$) Assume that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Then from the above computation, we have $(x, y) + (y, x) = 0$. Since, by assumption, V is a real vector space, $(x, y) = (y, x)$. Thus we have $2(x, y) = 0$, i.e., $\boxed{(x, y) = 0}$. Hence x and y are orthogonal. □

Proof of (b). Fix $x \neq 0$ and consider $y = ix$. Then by sesquilinearity,

$$(x, ix) + (ix, x) = -i(x, x) + i(x, x) = 0.$$

Thus $\|x + ix\|^2 = \|x\|^2 + \|ix\|^2$. However, $(x, ix) = i(x, x) = i\,\|x\|^2 \neq 0$ since $x \neq 0$. Thus x and ix are not orthogonal. So (a) is false for any complex inner product space V. □

Problem 5.4.3

In each of the following, you are given a domain D and a function $f : D \to \mathbb{C}$. Determine whether f has an anti-derivative on D.

(a) $f(z) = e^{1/z}\,\mathrm{Log}(z)$ **where D is the complex plane with the origin and negative real axis removed.**

(b) $f(z) = \dfrac{1}{z^2 - 1}$ **where D consists of all points in $\mathbb{C}$ except for ± 1.**

(c) $f(z) = \exp\left(\dfrac{1}{z^2}\right)$**, where $D = \mathbb{C} \setminus \{0\}$.**

Notes and Comments

Proof of (a). $\boxed{\textbf{True}}$ The domain D is a simply-connected subset of $\mathbb{C} \setminus \{0\}$, so $\mathrm{Log}(z)$ is a branch of log on D. That is, $\mathrm{Log}(z)$ is analytic on D. As $e^{1/z}$ is analytic on $\mathbb{C} \setminus \{0\} \supset D$, f is analytic on D. By the Global Cauchy Theorem, f has anti-derivative on D. □

Proof of (b). $\boxed{\textbf{False}}$ Consider a circular path around one of the poles (± 1). By the Residue Theorem, the integral depends on the index of the path. That is, reversing orientations will give opposite values. Hence the path integrals of f are not path-independent and so f has no anti-derivative. □

Proof of (c). $\boxed{\textbf{False}}$ Same as part (b). The point $0 \in \mathbb{C}$ is a pole of f and so f has no anti-derivative. □

Problem 5.4.4

Consider $C[0,1]$ with the uniform norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Show that the linear map

$$V : C[0,1] \to C[0,1]$$

defined by the formula

$$V(f)(x) = \int_0^x f(t)dt$$

is a bounded linear operator with *no* eigenvalues.

Notes and Comments

Proof. To show that V is a bounded linear operator, we must find $M > 0$ such that $\|V(f)\|_\infty \le M\|f\|_\infty$ for all $f \in C[0,1]$.

For any $x \in [0,1]$, we know that

$$\left|\int_0^x f(t)dt\right| \le \int_0^x |f(t)|\,dt \le \int_0^x \|f\|_\infty\,dt = x\,\|f\|_\infty \le \|f\|_\infty = \|f\|_\infty\,.$$

Thus $\|V(f)\|_\infty = \sup\limits_{x\in[0,1]}\left|\int_0^x f(t)dt\right| \le \|f\|_\infty$. That is $\boxed{\|V\| \le 1}$ and so V is bounded.

Claim: V has no eigenvalues.

To the contrary, suppose λ is an eigenvalue of V. Then $\exists f \in C[0,1]$ with $f \ne 0$ such that $V(f) = \lambda f$. If $\lambda = 0$ then we have $V(f) = 0$. By the Fundamental Theorem of Calculus,

$$f(x) = V(f)'(x) = 0$$

for all $x \in [0,1]$. However, this means $f = 0$. ↯ So we may assume $\lambda \neq 0$.

Observe that $\lambda f(0) = V(f)(0) = \int_0^0 f(t)dt = 0$. Thus, as $\lambda \neq 0$, $\boxed{f(0) = 0}$. By the Fundamental Theorem of Calculus, $f = \dfrac{1}{\lambda}V(f)$ is differentiable and we have

$$\lambda f'(x) = V(f)'(x) = f(x).$$

Thus $f'(x) = \dfrac{1}{\lambda}f(x)$. This ODE has unique solution $f(x) = Ce^{\frac{1}{\lambda}x}$ for some $C \neq 0$. However, for any choice of C, $\boxed{f(0) = C \neq 0}$. By the above, we have a problem. ↯ Hence V has no eigenvalues. $\square$

Problem 5.4.5

Find the limit of each of the following sequences of integrals. Justify fully. (Here m denotes Lebesgue measure on $\mathbb{R}$.)

(a) $\displaystyle\lim_{n\to\infty}\int_{[0,\infty)} f_n\ dm$ **where** $f_n(x) = \dfrac{\sin(nx)}{n(1+x^2)}$**,**

(b) $\displaystyle\lim_{n\to\infty}\int_{[0,\infty)} f_n\ dm$ **where** $f_n(x) = e^{-\frac{x}{n}}\dfrac{1}{1+x}$**.**

Notes and Comments

Proof of (a). First notice that f_n converges pointwise to f where $f(x) = 0$ and each f_n is measurable (since each one is a quotient of continuous functions). To use the Dominated Convergence Theorem, we must provide an integrable function g such that $|f_n(x)| \leq g(x)$ for all $n \geq 1$ and all $x \in [0,\infty)$. We claim that $g(x) = \frac{1}{1+x^2}$ will do.

Indeed, g is integrable and, for $n \geq 1$, we have

$$|f_n(x)| = \frac{|\sin(nx)|}{|n(1+x)^2|} \leq \frac{1}{n(1+x)^2} \leq \frac{1}{(1+x)^2}.$$

Invoking the Dominated Converge Theorem, we find

$$\lim_{n\to\infty}\int_{[0,\infty)} f_n\ dm = \int_{[0,\infty)} f\ dm = \int_{[0,\infty)} 0\ dx = 0.$$

That is, $\boxed{\displaystyle\lim_{n\to\infty}\int_{[0,\infty)} f_n\ dm = 0}$. $\square$

Proof of (b). First notice that f_n converges pointwise to f where $f(x) = \dfrac{1}{1+x}$ and each f_n is measurable (since each one is a quotient of continuous functions). To use the Dominated Convergence Theorem, we must produce an integrable function g such that $|f_n(x)| \leq g(x)$ for all $n \geq 1$. We claim that $g(x) = e^{-x}$ will do.

Indeed, g is integrable on this domain ($\int_0^\infty e^{-x}dx = 1$) and, for $n \geq 1$ and $x \in [0,\infty)$, we have

$$|f_n(x)| = \left|e^{\frac{-x}{n}}\frac{1}{1+x}\right| \leq e^{-x}\frac{1}{1+x} \leq e^{-x}.$$

By the Dominated Convergence Theorem, the integral $\lim_{n\to\infty}\int_{[0,\infty)} f_n\ dm$ exists and we have

$$\lim_{n\to\infty}\int_{[0,\infty)} f_n\ dm = \int_{[0,\infty)} f(x)\ dx = \int_{[0,\infty)} \frac{1}{1+x}\ dx.$$

Since this integral exists, we can use the methods of calculus to calculate the value of the integral. That is,

$$\int_{[0,\infty)} \frac{1}{1+x}\ dx = \lim_{y\to\infty}\int_{[0,y]} \frac{1}{1+x}\ dx = \lim_{y\to\infty}\Big[\ln|1+x|\Big|_0^y = \lim_{y\to\infty}\ln(1+y).$$

Thus $\int_{[0,\infty)} \frac{1}{1+x}\ dx$ diverges to infinity. That is, $\boxed{\lim_{n\to\infty}\int_{[0,\infty)} f_n\ dm \to \infty}$. □

Problem 5.4.6

Let f, g be 2π-periodic (Lebesgue) measurable functions on $\mathbb{R}$. Let $f * g$ denote the (normalized) *convolution* **function**

$$f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)dt.$$

(a) Show that if (their restrictions) $f, g \in L^2[-\pi,\pi]$ then $f * g(x)$ exists and is bounded on $[-\pi,\pi]$, in fact,

$$||f * g||_\infty = \sup_{x\in[-\pi,\pi]}|f * g(x)| \leq \frac{1}{2\pi}||f||_2||g||_2.$$

(b) Show that $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$ for all $n \in \mathbb{Z}$, where

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

is the n-th Fourier coefficient of f for $n \in \mathbb{Z}$.

Notes and Comments

Proof of (a). The fact that $f * g(x)$ exists and is bounded follows directly from the inequality. Indeed, $f * g(x)$ is bounded if and only if $\sup_{x\in[-\pi,\pi]}|f * g(x)|$ is finite. and the inequality produces a finite upper bound on this quantity. Moreover, $f * g(x)$ exists because the value of the integral is bounded on $[-\pi,\pi]$.

Now we'll show the inequality holds.[5] Define the auxiliary function $h_x(t) = g(x-t)$ for any fixed $x \in [-\pi,\pi]$. Then, using u-substitution ($u(t) = x - t$),

$$||h_x||_2^2 = \int_{-\pi}^{\pi}|h_x(t)|^2dt = -\int_{x+\pi}^{x-\pi}|g(t)|^2dt = \int_{-\pi}^{\pi}|g(t)|^2dt = ||g||_2^2,$$

where the third inequality follows by flipping the bounds of integration and because the function g is 2π-periodic. Thus

$$\sup_{x\in[-\pi,\pi]}\left|\int_{-\pi}^{\pi} f(t)g(x-t)dt\right| = \sup_{x\in[-\pi,\pi]}\left|\int_{-\pi}^{\pi} f(t)h_x(t)dt\right| \overset{(CS-\leq)}{\leq} ||f||_2||h_x||_2.$$

[5]The equality is just the definition of the supremum norm.

Putting this all together, we have

$$||f*g||_\infty = \frac{1}{2\pi}\sup_{x\in[-\pi,\pi]}\left|\int_{-\pi}^{\pi} f(t)g(x-t)dt\right| \le \frac{1}{2\pi}||f||_2||h_x||_2 = \frac{1}{2\pi}||f||_2||g||_2.$$

Thus the desired inequality holds and the result follows. □

Proof of (b). We have

$$\widehat{f*g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)dt\right)e^{-inx}dx.$$

We showed in part (a) that the interior integral is bounded, so we may apply Fubini's Theorem to switch order of integration. This gives

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t)g(x-t)e^{-inx}dxdt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x-t)e^{-in(x-t)}dx\right)dt.$$

To determine the value of the interior integral, we use u-substitution and the 2π-periodicity of g:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x-t)e^{-in(x-t)}dx = \frac{1}{2\pi}\int_{-\pi-t}^{\pi-t} g(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx = \hat{g}(n).$$

Putting the pieces together and collapsing the second integral, we now have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x-t)e^{-in(x-t)}dx\right)dt = \hat{g}(n)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt\right) = \widehat{f}(n)\widehat{g}(n).$$

From the first line, we can bring this all together to obtain $\boxed{\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)}$. □

Analysis Exam

Summer 2014

Problem 5.5.1

Let $(X, \mathcal{M})$ be a measurable space.

(a) **Let $\{f_n\}$ be a sequence of measurable functions with $f_n \colon X \to [-\infty, \infty]$. Show that the function $g \colon X \to [-\infty, \infty]$ defined by**

$$g(x) = \sup\{f_n(x) \ : \ n \geq 1\}$$

is measurable.

(b) **For $\{f_n\}$ as in part (a), show that the function $h \colon X \to [-\infty, \infty]$ defined by**

$$h(x) = \limsup\{f_n(x) \ : \ n \geq 1\}$$

is measurable.

(c) **Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be measurable functions. Let**

$$E = \{x \in X : f(x) > g(x)\}.$$

Starting with the definition of a measurable function, show that $E \in \mathcal{M}$.

Notes and Comments

Proof of (a). It suffices to show that $g^{-1}((a, \infty])$ is a measurable set for every $a \in \mathbb{R}$. Observe that

$$g^{-1}((a, \infty])) = \{x : g(x) > a\} = \{x : f_n(x) > a \text{ for some } n\} = \bigcup_n f_n^{-1}((a, \infty]),$$

which is a countable union of measurable sets and thus is a measurable set. $\square$

Proof of (b). Define $h_m \colon X \to [-\infty, \infty]$ by

$$h_m(x) = \sup_{n \geq m}\{f_n(x)\}.$$

By (a), each h_m is measurable. For each x, $\{h_m(x)\}$ is a non-increasing sequence, so

$$h(x) = \lim_{m \to \infty} h_m(x) = \inf_m\{h_m(x)\},$$

which is measurable by the analog to part (a) (replacing "sup" with "inf"). $\square$

Proof of (c). By the definition of a measurable function, $f^{-1}((a, \infty])$ and $g^{-1}([\infty, a))$ are measurable sets for every $a \in \mathbb{R}$. Then $f^{-1}((a, \infty]) \cap g^{-1}([\infty, a))$ is a measurable set, so the countable union

$$E' = \bigcup_{q \in \mathbb{Q}} f^{-1}((q, \infty]) \cap g^{-1}([-\infty, q))$$

is a measurable set. We claim that $E = E'$.

Let $x \in X$. Then, by the density of $\mathbb{Q}$ in $\mathbb{R}$,

$$\begin{aligned} x \in E \iff f(x) > g(x) &\iff \exists q \in \mathbb{Q}, f(x) > q > g(x) \\ &\iff \exists q \in \mathbb{Q}, x \in f^{-1}((q, \infty]) \text{ and } x \in g^{-1}([-\infty, a)) \\ &\iff x \in E'. \end{aligned}$$

Thus $E = E'$ and so $E \in \mathcal{M}$. $\square$

Problem 5.5.2

Let $(X, \mathcal{M}, \mu)$ be a measure space, let $\mathcal{N}$ be a σ-algebra on a set Y, and let $f\colon X \to Y$ be an $(\mathcal{M}, \mathcal{N})$-measurable function. Define $\nu\colon \mathcal{N} \to [0, \infty]$ by

$$\nu(A) = \mu(f^{-1}(A)).$$

(a) Show that ν is a measure on $(Y, \mathcal{N})$.

(b) For $g \in L^+(Y, \mathcal{N})$ (i.e., $g : Y \to [0, \infty]$ is a measurable function), show that

$$\int_Y g \; d\nu = \int_X g \circ f \; d\mu.$$

(Suggestion: First verify the statement when g is the characteristic function of a measurable set.)

Notes and Comments

Proof of (a). Notice that, since μ is a measure, we have

- $\nu(\varnothing) = \mu(f^{-1}(\varnothing)) = \mu(\varnothing) = 0$
- If $\{E_n\}$ is a countable collection of pairwise disjoint sets in $\mathcal{N}$, then

$$\nu\left(\bigcup_n E_n\right) = \mu\left(f^{-1}\left(\bigcup_n E_n\right)\right) = \mu\left(\bigcup_n f^{-1}(E_n)\right) \overset{(\star)}{=} \sum_n \mu(f^{-1}(E_n)) = \sum_n \nu(E_n).$$

where equality $(\star)$ holds because the sets $\{f^{-1}(E_n)\}$ are pairwise disjoint (as $\{E_n\}$ are pairwise disjoint) and f is $(\mathcal{M}, \mathcal{N})$-measurable.

□

Proof of (b). Let E be a measurable set in Y. Then

$$\chi_{f^{-1}(E)}(x) = \begin{cases} 1 & \text{if } x \in f^{-1}(E) \\ 0 & \text{if } x \notin f^{-1}(E) \end{cases} = \begin{cases} 1 & \text{if } f(x) \in E \\ 0 & \text{if } f(x) \notin E \end{cases} = \chi_E(f(x)),$$

so $\boxed{\chi_{f^{-1}(E)} = \chi_E \circ f}$. Then

$$\int_Y \chi_E \; d\nu = \nu(E) = \mu(f^{-1}(E)) = \int_X \chi_{f^{-1}(E)} \; d\mu = \int_X (\chi_E \circ f) \; d\mu.$$

Thus $\boxed{\int_Y \chi_E \; d\nu = \int_X \chi_E \circ f \; d\mu (\star)}$ for any measurable set $E \in \mathcal{N}$.

Now let ϕ be a simple function, i.e., $\phi = \sum_{i=1}^n a_i \chi_{E_i}$ where $a_i \in (0, \infty]$ and $E_i \in \mathcal{N}$. Then

$$\int_Y \phi \; d\nu = \sum_{i=1}^n a_i \int_Y \chi_i \; d\nu \overset{(\star)}{=} \sum_{i=1}^n a_i \int_X (\chi_i \circ f) \; d\mu = \int_X \left(\sum_{i=1}^n a_i \left(\chi_i \circ f\right)\right) d\mu = \int_X (\phi \circ f) \; d\mu.$$

Thus $\boxed{\int_Y \phi \, d\nu = \int_X \phi \circ f \, d\mu(\star\star)}$ for any simple function ϕ.

Finally let $g\colon Y \to [0, \infty]$ be measurable. Then g is a pointwise limit of a sequence of simple functions $\phi_1 \le \phi_2 \le \cdots$. By the Monotone Convergence Theorem,

$$\int_Y g \, d\nu = \lim_{i\to\infty} \int_Y \phi_i \, d\nu \overset{(\star\star)}{=} \lim_{i\to\infty} \int_X (\phi_i \circ f) \, d\mu.$$

Now $\phi_i \le \phi_{i+1}$, so $\phi_i \circ f \le \phi_{i+1} \circ f$. So we can apply the Monotone Convergence Theorem again to obtain

$$\lim_{i\to\infty} \int_X (\phi_i \circ f) \, d\mu = \int_X \left(\lim_{i\to\infty} (\phi_i \circ f) \right) d\mu = \int_X g \circ f \, d\mu.$$

Hence $\boxed{\int_Y g \, d\nu = \int_X g \circ f \, d\mu}$ as desired. $\square$

Problem 5.5.3

Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function.

(a) Prove that if f is nonconstant, then the image of f is dense in $\mathbb{C}$.

(b) Suppose $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

Notes and Comments

Proof of (a). Let $w \in \mathbb{C}$ and suppose w is not already in the image of f. We will prove that the image of f has points arbitrarily close to w and so f has dense image.

Let $\varepsilon > 0$. Since $f(z) \neq w$ for all $z \in \mathbb{C}$, the function $\frac{1}{f(z)-w}$ is defined on $\mathbb{C}$, entire, and nonconstant. By Liouville's Theorem, $\frac{1}{f(z)-w}$ is unbounded and so there is z_0 such that $\left|\frac{1}{f(z_0)-w}\right| > \frac{1}{\varepsilon}$. Thus $|f(z_0) - w| < \varepsilon$. $\square$

Proof of (b). Since f is entire, its Taylor series at 0 converges to f everywhere. That is $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then

$$f(1/z) = \sum_{n=0}^{\infty} c_n z^{-n} \tag{5.1}$$

for all $z \in \mathbb{C} \setminus \{0\}$. As $\lim_{z\to 0} f(1/z) = \lim_{z\to\infty} f(z) = \infty$, we know $f(1/z)$ has a pole at 0. Consequently the Laurent series of $f(1/z)$ at 0 has only finitely many terms of negative degree. From (5.1), we see that the Laurent series of $f(1/z)$ has no terms of positive degree. Hence, the Laurent series is a finite sum $f(1/z) = \sum_{n=0}^{d} c_n z^{-n}$ for some $d < \infty$. Thus $\boxed{f(z) = \sum_{n=0}^{d} c_n z^n}$ and so f is a polynomial. $\square$

Problem 5.5.4

Let $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ be normed vector spaces. Equip the vector space direct product $V \times W$ with the norm $||(x, y)||_1 = ||x||_V + ||y||_W$ for $x \in V$ and $y \in W$. Suppose that $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are Banach spaces. Prove that $(V \times W, ||\cdot||_1)$ is also a Banach space.

Notes and Comments

Proof. Observe that $(V \times W, ||\cdot||_1)$ is a normed vector space. Thus we need only show that $(V \times W, ||\cdot||_1)$ is complete. Let $\{(v_i, w_i)\}_{i=1}^{\infty}$ be a Cauchy sequence in $V \times W$. We first claim that the component sequences are Cauchy. Let $\varepsilon > 0$ be arbitrary. Then there exists $N \in \mathbb{Z}$ such that, for all $m, n \geq N$, $\boxed{||(v_m, w_m) - (v_n, w_n)||_1 < \varepsilon}$. For $m, n \geq N$, using the definition of the norm, we see that

$$||v_m - v_n||_V \leq ||v_m - v_n||_V + ||w_m - w_n||_W = ||(v_m, w_m) - (v_n, w_n)||_1 < \varepsilon.$$

Thus $\{v_i\}_{i=1}^{\infty}$ is a Cauchy sequence in V. Since V is Banach, $\exists v \in V$ such that $\{v_i\}_{i=1}^{\infty}$ converges to v. Analogously, $\exists w \in W$ such that $\{w_i\}_{i=1}^{\infty}$ converges to w. We claim that $\{(v_i, y_i)\}_{i=1}^{\infty}$ converges to (v, w).

Let $\varepsilon > 0$ be arbitrary. Then there exists a $N_v \in \mathbb{Z}$ such that, for all $m, n \geq N_v$, $\boxed{||v_m - v_n||_V < \frac{\varepsilon}{2}}$. Similarly, there exists a $N_w \in \mathbb{Z}$ such that, for all $m, n \geq N_w$, $\boxed{||w_m - w_n||_W < \frac{\varepsilon}{2}}$. Take $N = \max(N_v, N_w)$ and compute

$$||(v_m, w_m) - (v_n, w_n)||_1 = ||v_m - v_n||_V + ||w_m - w_n||_W < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

Hence the original sequence converges to (v, w). Thus $V \times W$ is complete and hence Banach as desired. □

Problem 5.5.5

Consider the real Hilbert space $L^2(0, 1)$ with respect to Lebesgue measure on the open unit interval $(0, 1)$. For each $f \in L^2(0, 1)$, define $M(f) : (0, 1) \to \mathbb{R}$ by

$$M(f)(x) = x\, f(x).$$

- **(a) Show that M is a well-defined bounded linear operator on $L^2(0, 1)$.**
- **(b) Show that M is injective and M is *self-adjoint*, i.e., $(M(f), g) = (f, M(g))$ for all $f, g \in L^2(0, 1)$.**

Notes and Comments

Proof of (a). To show M is well-defined, we need to show two things: (i) if $f = g$ almost everywhere, then $M(f) = M(g)$ almost everywhere; and (ii) if $f \in L^2(0, 1)$, then $M(f) \in L^2(0, 1)$.

(i) Suppose $f = g$ everywhere except a set N of measure zero. Then, for $x \in (0, 1) \setminus N$, we have $x\, f(x) = x\, g(x)$, so $M(f) = M(g)$ everywhere except N.

(ii) Suppose $f \in L^2(0, 1)$, meaning that f is Lebesgue-measurable and $\int_0^1 |f(x)|^2\, dx < \infty$. Products of measurable functions are measurable, so $M(f)(x) = x\, f(x)$ is Lebesgue-measurable. As $|x| \leq 1$ on $(0, 1)$, we have $|x\, f(x)| \leq |f(x)|$ on $(0, 1)$ and so

$$\int_0^1 |x\, f(x)|^2\, dx \leq \int_0^1 |f(x)|^2\, dx < \infty. \tag{5.2}$$

Therefore $M(f) \in L^2(0, 1)$

Therefore, M is well-defined.

Next, we show that M is a bounded linear map:

- Linearity: $M(\lambda f + \mu g) = x\,(\lambda f + \mu g) = \lambda x f + \mu x g = \lambda\, M(f) + \mu\, M(g)$.
- Bounded: By (5.2), $\|M(f)\|_2 \le \|f\|_2$ for all $f \in L^2(0,1)$, so $\|M\| \le 1$.

□

Proof of (b). If $M(f) = 0$, then $x\, f(x) = 0$ for almost all $x \in (0,1)$. Then $f(x) = 0$ for almost all $x \in (0,1)$, meaning $f = 0$ in $L^2(0,1)$. Therefore M is injective.

For $f, g \in L^2(0,1)$, we have

$$(M(f), g) = \int_0^1 (x\, f(x))\, g(x)\; dx = \int_0^1 (x\, f(x))\, g(x)\; dx = \int_0^1 f(x)\, (x\, g(x))\; dx = (f, M(g)).$$

Therefore M is self-adjoint. □

Problem 5.5.6

Let $(V, \|\cdot\|)$ be a normed vector space over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). Let $M \subsetneq V$ be a proper closed subspace of V and let $x \in V \setminus M$.

(a) Show that $\delta = \inf\{\|x-y\| \;:\; y \in M\} > 0$.

(b) Show that there exists a bounded linear functional $f \in V^*$ such that $\|f\| = 1$ and $f(x) = \delta$ and $f|_M = 0$. (Hint: Work with $M + \mathbb{F}x$.)

Notes and Comments

Proof of (a). Suppose $\delta = 0$. Then, for each $n \in \mathbb{N}$, there is an element $y_n \in M$ such that $\|x - y_n\| < 1/n$. Then $\lim_{n\to\infty} \|x - y_n\| = 0$, meaning that the sequence $\{y_n\}$ converges to x. But M is closed, so M contains its limit points and thus $x \in M$. ↯ □

Proof of (b). First define $f\colon (M + \mathbb{F}x) \to \mathbb{F}$ as follows: given $y + \lambda x \in M + \mathbb{F}x$, define

$$f(y + \lambda x) = \lambda\delta.$$

This is well-defined because $x \notin M$ so $M + \mathbb{F}x$ is a direct sum of vector spaces. Clearly f is linear, $f(x) = \delta$, and $f|_M = 0$. Now we show that $\|f\| = 1$.

Let $y + \lambda x \in M + \mathbb{F}x$. If $\lambda = 0$, then $|f(y + \lambda x)| = 0 \le \|y + \lambda x\|$. If $\lambda \ne 0$, then $-(1/\lambda)y \in M$, so $\|x - (-1/\lambda)y\| \ge \delta$. By the definition of δ,

$$|f(y + \lambda x)| = |\delta\lambda| = |\lambda|\,\delta \le |\lambda|\,\|x - (-1/\lambda)y\| = \|y + \lambda x\|.$$

This shows that $\boxed{\|f\| \le 1}$. In particular, f is a bounded linear operator.

Now let $\varepsilon > 0$. Then there is $y \in M$ such that $\|x - y\| \le \delta + \varepsilon$. Then

$$\|f\| \ge \frac{|f(x-y)|}{\|x-y\|} = \frac{\delta}{\|x-y\|} \ge \frac{\delta}{\delta + \varepsilon}.$$

Therefore $\|f\| \geq \frac{\delta}{\delta+\varepsilon}$ for all $\varepsilon > 0$. Now we let $\varepsilon \to 0$ and, since $\delta > 0$ by part (a), we conclude that $\boxed{\|f\| \geq 1}$. Therefore $\boxed{\|f\| = 1}$ as desired.

Finally, by the Hahn–Banach theorem, f can be extended to a bounded linear functional $\widetilde{f} : V \to \mathbb{F}$ with $\|f\| = 1$. Since $\widetilde{f}$ is an extension of f, we still have $\widetilde{f}(x) = \delta$ and $\widetilde{f}\Big|_M = 0$. □

Analysis Exam

Fall 2014

Problem 5.6.1

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence on $\mathcal{M}$ such that $E_1 \subset E_2 \subset E_3 \subset \dots$ and let $E = \bigcup_{n=1}^{\infty} E_n$. Also let $\{A_n\}_{n=1}^{\infty}$ be a sequence on $\mathcal{M}$ such that $A_1 \supset A_2 \supset A_3 \supset \dots$ and let $A = \bigcap_{n=1}^{\infty} A_n$.

(a) Suppose that $f : X \to \mathbb{R}$ is integrable. Show that

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_{E_n} f \, d\mu \tag{5.3}$$

and

$$\int_A f \, d\mu = \lim_{n\to\infty} \int_{A_n} f \, d\mu. \tag{5.4}$$

(b) Suppose that $f \in L^+(X, \mathcal{M}, \mu)$, i.e., f is a non-negative measurable real-valued function but it is not necessarily integrable. Show that Equation (5.3) is valid for f.

(c) Give an example of a measure space $(X, \mathcal{M}, \mu)$ and a function $f \in L^+(X, \mathcal{M}, \mu)$ such that Equation (5.4) fails to hold for f.

Notes and Comments

Proof of (a). Define $f_n = f \cdot \chi_{E_n}$. Then $f_n \to f \cdot \chi_E$ and, since f is integrable,

$$\int_X |f_n| \, d\mu = \int_{E_n} |f| \, d\mu \le \int_X |f| \, d\mu < \infty.$$

So $f_n \in L^1(X)$, i.e., f_n is integrable. Also we have $|f_n| \le |f|$. Since f is integrable, we know that $|f|$ is a positive integrable function. Thus, by the Dominated Convergence Theorem $(*)$,

$$\int_E f \, d\mu = \int_X f \cdot \chi_E \, d\mu \overset{(*)}{=} \lim_{n\to\infty} \int_X f \cdot \chi_{E_n} \, d\mu = \lim_{n\to\infty} \int_{E_n} f \, d\mu.$$

Hence $\boxed{\int_E f \, d\mu = \lim_{n\to\infty} \int_{E_n} f \, d\mu}$.

Similarly, define $g_n = f \cdot \chi_{A_n}$. Then $g_n \to f \cdot \chi_A$ and, as above, $g_n \in L^1(X)$ and $|g_n| \le |f|$. By the Dominated Convergence Theorem,

$$\int_A f \, d\mu = \int_X f \cdot \chi_A \, d\mu \overset{(*)}{=} \lim_{n\to\infty} \int_X f \cdot A_n \, d\mu = \lim_{n\to\infty} \int_{A_n} f \, d\mu.$$

Hence $\boxed{\int_A f \, d\mu = \lim_{n\to\infty} \int_{A_n} f \, d\mu}$. □

Proof of (b). Since $f \in L^+(X)$, we have $f_n = f \cdot \chi_{E_n} \in L^+(X)$. Since the E_n's form a chain, $f_n \leq f_{n+1}$. By the Monotone Convergence Theorem $(*)$, $\lim\limits_{n\to\infty} f_n = f \cdot \chi_E$ exists and, more importantly,

$$\int_E f\ d\mu = \int_X f \cdot \chi_E\ d\mu \overset{(*)}{=} \lim_{n\to\infty} \int_X f_n\ d\mu = \lim_{n\to\infty} \int_{E_n} f\ d\mu.$$

That is, $\boxed{\int_E f\ d\mu = \lim_{n\to\infty} \int_{E_n} f\ d\mu}$. □

Proof of (c). Consider the measure space $(\mathbb{R}, \mathcal{L}, \mu)$ where μ is Lebesgue measure. Define $f = \chi_{(0,\infty)}$, the characteristic function on $(0, \infty)$. Then $f \in L^+(\mathbb{R})$.

Choose $A_n = [n, \infty)$ for $n \in \mathbb{N}$, a nested sequence of sets. Then $A = \varnothing$ and so

$$\int_A f\ d\mu = 0.$$

However,

$$\int_{A_n} f\ d\mu = \int_n^\infty 1\ d\mu = \infty.$$

Thus $\boxed{\int_A f\ d\mu = 0 \neq \infty = \lim_{n\to\infty} \int_{A_n} f\ d\mu.}$. □

Problem 5.6.2

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space satisfying $\mu(X) < \infty$. Show that $L^q(X, \mathcal{M}, \mu) \subset L^p(X, \mathcal{M}, \mu)$ whenever $0 < p < q$. (Be sure to include the case that $q = \infty$.)

Notes and Comments

Proof. First, we will do away with the case $q = \infty$.[6] Assume $f \in L^\infty(X)$. Then $\exists M \geq 1$ so that $\|f\|_\infty \leq M$. That is, $|f(x)| \leq M$ a.e. Then since $M \geq 1$, we have $M^p \geq |f|^p$ and so

$$\|f\|_p^p = \int_X |f|^p\ d\mu \leq \int_X M^p\ d\mu = M^p \mu(X) < \infty.$$

Now assume $q < \infty$. Define $S_1 = \{x \in X \mid |f(x)| \geq 1\}$ and $S_2 = \{x \in X \mid |f(x)| < 1\}$. Then S_1 and S_2 partition X into disjoint $\mathcal{M}$-measurable sets[7] and, since $p < q$,

$$\begin{aligned}\|f\|_p^p = \int_X |f|^p\ d\mu &= \int_{S_1} |f|^p\ d\mu + \int_{S_2} |f|^p\ d\mu \\ &\leq \int_{S_1} |f|^q + \int_{S_2} 1\ d\mu \leq \int_X |f|^q\ d\mu + \int_X 1\ d\mu = \|f\|_q^q + \mu(X) < \infty.\end{aligned}$$

That is, $f \in L^p(X)$ in either case. □

[6] As the largest "number," it gets special treatment.

[7] We can rewrite S_1 and S_2 as the preimages of unions of intervals. Specifically, as one graduate student commented, $S_2 = f^{-1}((-1, 1))$ and S_1 is the preimage of "all the other crap."

Problem 5.6.3

Let Γ_R^+ be the semicircle defined by $|z| = R$ and $\mathrm{Im}(z) \geq 0$, and let Γ_R^- be the semicircle given by $|z| = R$ and $\mathrm{Im}(z) \leq 0$. Give both semicircles the counterclockwise orientation.

(a) Evaluate

$$\lim_{R\to\infty} \int_{\Gamma_R^+} \frac{e^{iz}}{z^4} dz$$

(b) Evaluate

$$\lim_{R\to\infty} \int_{\Gamma_R^-} \frac{e^{iz}}{z^4} dz$$

Notes and Comments

Proof of (a). We claim that this limit of integrals is zero. Parametrize Γ^+ as $Re^{i\theta}$ for $0 \leq \theta \leq \pi$. This allows us to rewrite the desired integral (for fixed R) as

$$\int_0^\pi \frac{e^{iR(\cos\theta + i\sin\theta)}}{(Re^{i\theta})^4} iRe^{i\theta} d\theta.$$

Consider the absolute value of this integral:

$$\begin{aligned}\left| \int_0^\pi \frac{e^{iR(\cos\theta + i\sin\theta)}}{(Re^{i\theta})^4} iRe^{i\theta} d\theta \right| &\leq \int_0^\pi \left| \frac{e^{iR(\cos\theta + i\sin\theta)}}{(Re^{i\theta})^4} iRe^{i\theta} \right| d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{R^3} d\theta.\end{aligned}$$

Since $0 \leq \theta \leq \pi$, we have $\sin\theta > 0$ and so the numerator decays exponentially. Let $f_n(\theta) = \dfrac{e^{-n\sin\theta}}{n^3}$ and $g(\theta) = 1$ (which is integrable and dominates f_n). Then $f_n \to f$ pointwise where $f(\theta) = 0$. By the Dominated Convergence Theorem, we have

$$\lim_{R\to\infty} \int_0^\pi \frac{e^{-R\sin\theta}}{R^3} d\theta = \lim_{n\to\infty} \int_0^\pi f_n(\theta) d\theta = \int_0^\pi 0 d\theta = 0.$$

Thus the limit of the integrals is 0, as desired. □

Proof of (b). We begin by noting that we can write down the Laurent series for $f(z) = \dfrac{e^{iz}}{z^4}$ using the known series for e^z to obtain $f(z) = \frac{1}{z^4} + \frac{i}{z^3} - \frac{1}{2z^2} - \frac{i}{6z} + \frac{1}{24} + O(z)$. This allows us to determine that the residue of f at 0 is $\frac{-i}{6}$ and hence the integral of f over the entire circle of radius R is $2\pi i \cdot \frac{-i}{6} = \frac{\pi}{3}$. Since we know that the entire integral is the sum of the individual integrals over the two semicircles, and we know that the upper integral is zero from part (a), the bottom semicircle must give the entire value of $\frac{\pi}{3}$ from the Residue Theorem. □

Problem 5.6.4

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces. Give the Cartesian product

$$V \times W = \{(x, y) \mid x \in V, y \in W\}$$

the obvious coordinate-wise defined vector space structure and the norm

$$\|(x, y)\|_{V\times W} = \|x\|_V + \|y\|_W \text{ for } x \in V, y \in W.$$

Prove that the graph $G(T) = \{(x, T(x)) \mid x \in V\}$ of a continuous linear mapping $T : V \to W$ is a closed, linear subspace of $(V \times W, \|\cdot\|_{V\times W})$.

Notes and Comments

Proof. First, we deal with the linear subspace issue. By the linearity of T,

$$\alpha(x, T(x)) + \beta(y, T(y)) = (\alpha x + \beta y, \alpha T(x) + \beta T(y)) = (\alpha x + \beta y, T(\alpha x + \beta y)).$$

Thus $\boxed{G(T) \text{ is a subspace}}$.

Now suppose that $\lim_{n\to\infty}(x_n, T(x_n)) = (x, y)$. Then

$$\|x_n - x\|_V + \|T(x_n) - y\|_W = \|(x_n, T(x_n)) - (x, y)\|_{V\times W} \to 0.$$

That is, $\|x_n - x\|_V \to 0$ and $\|T(x_n) - y\|_W \to 0$. Thus $x_n \to x$. By the continuity of T, we have

$$y = \lim_{n\to\infty} T(x_n) = T\left(\lim_{n\to\infty} x_n\right) = T(x).$$

Thus $(x, y) = (x, T(x)) \in G(T)$. So $\boxed{G(T) \text{ is closed}}$. □

Problem 5.6.5

Let c denote the $\mathbb{C}$-vector space of all convergent complex sequences. Show that c is a Banach space when equipped with the supremum norm from ℓ^∞:

$$\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

Notes and Comments

Proof. Let $(a_n)_{n=1}^\infty$ be a Cauchy sequence in c. Then $(a_{n,i})_{n=1}^\infty$ is Cauchy in $\mathbb{C}$. By the completeness of $\mathbb{C}$, we know that $a_{n,i} \to x_i$ for some $x_i \in \mathbb{C}$. Define $\boxed{x = (x_1, x_2, \dots)}$.

Let $\varepsilon > 0$. Since (a_n) is Cauchy, $\exists N$ such that for all $n, m \geq N$, we know $\|a_n - a_m\|_\infty < \varepsilon$. So for $n \geq N$, we have

$$\begin{aligned} \|a_n - x\|_\infty &= \sup_{i\in\mathbb{N}} |a_{n,i} - x_i| \\ &= \sup_{i\in\mathbb{N}} \left|a_{n,i} - \lim_{m\to\infty} a_{m,i}\right| \\ &= \sup_{i\in\mathbb{N}} \lim_{m\to\infty} |a_{n,i} - a_{m,i}|. \end{aligned}$$

For $m \geq N$, we know $|a_{n,i} - a_{m,i}| \leq \|a_n - a_m\|_\infty < \varepsilon$ by definition of the supremum norm and N. Thus we have $\|a_n - x\|_\infty < \varepsilon$. Hence $a_n \to x$, i.e., c is complete. As c is a complete vector subspace of ℓ^∞, c is a Banach space. □

Problem 5.6.6

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Consider the set

$$F(\mathcal{H}) = \{T \in B(\mathcal{H}) : \dim(\text{range}(T)) < \infty\}$$

of bounded *finite rank* **operators. It is easy to see that $F(\mathcal{H})$ is a subalgebra of the algebra $B(\mathcal{H})$, but it has further structure.**

(a) Show that if $T \in F(\mathcal{H})$ then $T^* \in F(\mathcal{H})$ and $\dim(\text{range}(T^*)) = \dim(\text{range}(T))$.

(b) Show that if $T \in F(\mathcal{H})$ then $ST, TS \in F(\mathcal{H})$ for all $S \in B(\mathcal{H})$.

Notes and Comments

Proof of (a). Assume T is of finite rank. Then $T|_{(\ker T)^\perp} : (\ker T)^\perp \to \text{range}(T)$ is an injective map into a finite-dimensional vector space. That is, $T|_{(\ker T)^\perp}$ is an isomorphism.

As $\text{range}(T^*) = \ker(T)^\perp \cong \text{range}(T)$, we immediately obtain both conclusions. More explicitly, $T^* \in F(\mathcal{H})$ and its range has the same dimension. □

Proof of (b). Let $\{T(v_1), \dots, T(v_k)\}$ be a basis for the range of T. Then $\text{range}(TS) \subseteq \text{range}(T)$ and so $\boxed{TS \in F(\mathcal{H})}$.

We also have $\{S(T(v_i))\}$ spanning $\text{range}(ST)$ and so $\boxed{ST \in F(\mathcal{H})}$. □

Analysis Exam

Summer 2015

Problem 5.7.1

Suppose that $\{f_n\}$ is a sequence of analytic functions converging uniformly on compact sets of a domain D to a function f. Prove that f is analytic on D.

Notes and Comments

Proof. See the solution to problem 2(a) from Summer 2012 (5.1.2). □

Problem 5.7.2

Let f be an entire function such that

$$\lim_{z\to\infty}\left|\frac{f(z)}{z^2}\right| = L < \infty$$

exists. Prove that there are complex constants a, b and c such that $f(z) = az^2 + bz + c$.

Notes and Comments

Proof. Since f is entire, $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in \mathbb{C}$. Thus

$$g(z) = \frac{f(z)}{z^2} = c_0 z^{-2} + c_1 z^{-1} + h(z)$$

where $h(z)$ is entire. Since

$$\lim_{z\to\infty} |g(z)| = L,$$

$h(z)$ must be bounded near ∞. That is, h is bounded on some disk $\{z \in \mathbb{C} \mid |z| \geq R\}$ for some $R > 0$. As $h(z)$ is entire, it is also bounded on $\{z \in \mathbb{C} \mid |z| \leq R\}$. Hence h is bounded.

Thus, as h is a bounded entire function, it must be constant by Liouville's Theorem. Hence

$$g(z) = c_0 z^{-2} + c_1 z^{-1} + c_2.$$

That is, $\boxed{f(z) = c_0 + c_1 z + c_2 z^2}$. □

Problem 5.7.3

Recall that Lebesgue measure m on $\mathbb{R}$ has the property that for every measurable set E, we have

$$m(E) = \inf\{m(V) : E \subset V \text{ and } V \text{ is open in } \mathbb{R}\}. \tag{5.5}$$

Given a measurable set E and $\varepsilon > 0$, show that there is a closed set F and an open set V such that $F \subset E \subset V$ and $m(V \setminus F) < \varepsilon$. Be careful: it is not immediate from (5.5) that there is an open set V such that $m(V \setminus E) < \varepsilon$ unless $m(E) < \infty$.

Notes and Comments

Proof. See the solution to problem 5 from Fall 2012 (5.2.5). □

Problem 5.7.4

Let E and F be normed linear spaces over $\mathbb{C}$.

(a) State the definitions of a bounded linear map between E and F and of the dual space E^*.

(b) Give an example of a linear map defined everywhere on a normed linear space but that is not bounded.

(c) Let $T : E \to F$ be a linear map such that for all $\varphi \in F^*$, we have $\varphi \circ T \in E^*$. Prove that T is bounded.

Notes and Comments

Proof of (a). A linear map $T : E \to F$ is *bounded* if $\exists M \geq 0$ such that $\|T(x)\|_F \leq M \|x\|_E$ for all $x \in E$.

The dual space $E^* = \{\varphi : E \to \mathbb{C} \mid \varphi \text{ is a bounded linear functional}\}$. □

Proof of (b). Let $E = C^1[0,1]$ and $F = C[0,1]$, each equipped with $\|\cdot\|_\infty$.[8] Define $T : E \to F$ by $T(f) = \dfrac{df}{dx}$.

Since the derivative operator is a linear map, it suffices to show that T is not bounded. Indeed, consider $f_n(x) = e^{inx} \in C^1[0,1]$ for $n \in \mathbb{Z}_+$. Then $\|f_n\|_\infty = 1$. However, $T(f_n)(x) = ine^{inx}$ has norm $\|T(f_n)\|_\infty = n$ for each n. Hence T is not bounded. □

Proof of (c). Consider the map $S : F^* \to E^*$ given by $S(\varphi) = \varphi \circ T$.[9] We claim that S is a bounded linear map. Linearity is immediate. For boundedness, first notice that E^* and F^* are both Banach spaces.[10]

For $x \in E$ and $\varphi \in F^*$ with $\|\varphi\| = 1$, we have

$$|S(\varphi)(x)| = |\varphi(T(x))| \leq \|\varphi\| \, \|T(x)\|_F = \|T(x)\|_F < \infty.$$

Thus $\sup_{S(\varphi)} |S(\varphi)(x)| \leq \|T(x)\|_F < \infty$. By the Uniform Boundedness Principle (Banach–Steinhaus Theorem), we have

$$\|S\| = \sup \{\|S(\varphi)\| \mid \|\varphi\| = 1\} < \infty.$$

Thus S is bounded as desired.

Finally, we show that $\|S\| \geq \|T\|$.[11]

Let m be any number less than $\|T\|$. Then $\exists x \in E$ such that $\|T(x)\|_F \geq m \|x\|_E$. By the Hahn–Banach Theorem, $\exists \psi \in F^*$ such that $\|\psi\| = 1$ and $|\psi(T(x))| = \|T(x)\|_F$. That is,

$$|S(\psi)(x)| = |\psi(T(x))| = \|T(x)\|_F \geq m \|x\|_E .$$

Thus $\|S(\psi)\| \geq m$. As ψ is a unit (dual) vector, $\|S\| \geq m$. Hence the results follows by taking the supremum over $m < \|T\|$. □

[8] That is, E consists of functions $f : [0,1] \to \mathbb{C}$ with continuous first derivative and F consists of continuous functions $f : [0,1] \to \mathbb{C}$.

[9] It's very tempting to write T^* to denote a pullback map, but since we're in functional analysis, that would be horribly misinterpreted as the adjoint. Also, for point of reference, the map S is called the *transpose* of T.

[10] It is a general fact that V^* is a Banach space as, by definition, is the bounded linear maps $B(V, \mathbb{F})$.

[11] Since S is bounded, this implies that T must be as well.

Problem 5.7.5

Let $\mathcal{A}$ be a unital Banach algebra.

(a) Consider $a \in \mathcal{A}$ with $\|a\| < 1$. Show that $1_{\mathcal{A}} - a$ is invertible. Does the converse hold?

(b) Prove that the set of invertible elements in $\mathcal{A}$ is open.

(c) Let $a \in \mathcal{A}$ and assume that its spectrum $\mathrm{Sp}_{\mathcal{A}}(a)$ is not empty. Prove that $\mathrm{Sp}_{\mathcal{A}}(a)$ is compact.

Notes and Comments

Proof of (a). Since $\|a\| < 1$, we know that $\|a^n\| \leq \|a\|^n < 1$ since $\mathcal{A}$ is a Banach algebra. Thus

$$\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1 - \|a\|} < \infty.$$

Since $\mathcal{A}$ is complete, this means that $b := \sum_{n=0}^{\infty} a^n \in \mathcal{A}$. Now we note that $(1_{\mathcal{A}} - a)b = 1_{\mathcal{A}}$ since

$$(1_{\mathcal{A}} - a)b = \sum_{n=0}^{\infty} (a^n - a^{n+1}) = 1_{\mathcal{A}}.$$

Thus $1_{\mathcal{A}} - a$ is invertible as desired.

The converse, however, is decidedly false. Consider $\mathcal{A} = \mathbb{C}$ with the usual norm. Consider $a = 1 + i$. Then $1 - a = -i$ is invertible ($a^{-1} = i$). However, $\|a\| = \sqrt{2} > 1$. ☐

Proof of (b). Let $GL(\mathcal{A})$ denote the invertible elements of $\mathcal{A}$ and pick $a \in GL(\mathcal{A})$. Then $a^{-1} \in \mathcal{A}$ and $\|a^{-1}\| > 0$. Define $\varepsilon := \dfrac{1}{\|a^{-1}\|}$ and consider the ball $B_\varepsilon(a)$.

We will show that $B_\varepsilon(a) \subset GL(\mathcal{A})$. Indeed, let $b \in B_\varepsilon(a)$. Then, by definition, $\|b - a\| < \varepsilon$. Since $\mathcal{A}$ is a Banach algebra, we have

$$\left\|1_{\mathcal{A}} - a^{-1}b\right\| = \left\|a^{-1}(a - b)\right\| \leq \left\|a^{-1}\right\| \|a - b\| < \left\|a^{-1}\right\| \varepsilon = 1.$$

That is, by part (a), $a^{-1}b = 1_{\mathcal{A}} - (1_{\mathcal{A}} - a^{-1}b) \in GL(\mathcal{A})$. Since $GL(\mathcal{A})$ is a group, $b \in GL(\mathcal{A})$ as desired. Hence $GL(\mathcal{A})$ is open. ☐

Proof of (c). Let $a \in \mathcal{A}$ such that $\mathrm{Sp}_{\mathcal{A}}(a) \neq \varnothing$.

For $\lambda \in \mathrm{Sp}_{\mathcal{A}}(a)$, we have $|\lambda| \leq \|a\|$ and so $\mathrm{Sp}_{\mathcal{A}}(a) \subseteq \overline{B_{\|a\|}(0)}$. This follows from part (a). Indeed, if $\|a\| / |\lambda| < 1$ then $1_{\mathcal{A}} - \frac{a}{\lambda} \in GL(\mathcal{A})$. Hence $a - \lambda 1_{\mathcal{A}} \in GL(\mathcal{A})$.

Moreover, $\mathrm{Sp}_{\mathcal{A}}(a)$ is closed in $\mathbb{C}$.[12] Thus $\mathrm{Sp}_{\mathcal{A}}(a)$ is a closed subset of a compact set; hence compact. ☐

[12] It's the complement of the resolvent, an open subset of $\mathbb{C}$.

Problem 5.7.6

Let $\mathcal{H}$ be a Hilbert space, T in $\mathcal{B}(\mathcal{H})$ and T^* its adjoint.

(a) Prove that $\ker T = (\operatorname{ran} T^*)^\perp$.

(b) Let A be a subset of $\mathcal{H}$. Prove that $A^\perp = \overline{\operatorname{span}(A)}^\perp$.

(c) Show that T is injective if and only if T^* has dense range.

Notes and Comments

Proof of (a). Notice that

$$x \in \ker T \Leftrightarrow 0 = \langle T(x), y\rangle = \langle x, T^*(y)\rangle \ \forall y \in \mathcal{H} \Leftrightarrow x \in (\operatorname{ran} T^*)^\perp.$$

Thus $\ker T = (\operatorname{ran} T^*)^\perp$. □

Proof of (b). ($\subseteq$): Let $x \in A^\perp$. Then $\boxed{\langle x, a\rangle = 0}$ $(*)$ for all $a \in A$. Pick $y \in \overline{\operatorname{span}(A)}$. Then $\exists (y_n)_{n=1}^\infty \subset \operatorname{span}(A)$ such that $y_n \to y$. By conjugate linearity in the second component and $(*)$, $\boxed{\langle x, y_n\rangle = 0}$. Hence, by continuity of the inner product,

$$\langle x, y\rangle = \lim_{n\to\infty} \langle x, y_n\rangle = \lim_{n\to\infty} 0 = 0.$$

Thus $x \in \overline{\operatorname{span}(A)}^\perp$ as desired.

($\supseteq$): Let $x \in \overline{\operatorname{span}(A)}^\perp$. Note that $A \subset \operatorname{span}(A) \subset \overline{\operatorname{span}(A)}$. Thus $x \in A^\perp$.[13]

Thus $A^\perp = \overline{\operatorname{span}(A)}^\perp$ as desired. □

Proof of (c). By part (a) and (b), we have

$$\ker T \overset{(a)}{=} (\operatorname{ran} T^*)^\perp \overset{(b)}{=} \overline{\operatorname{span}(\operatorname{ran} T^*)}^\perp = \overline{\operatorname{ran} T^*}^\perp.$$

Thus $\boxed{\ker T = \overline{\operatorname{ran} T^*}^\perp}$.

If $\ker T = \{0\}$ then $\overline{\operatorname{ran} T^*}^\perp = \{0\}$. Thus $\boxed{\overline{\operatorname{ran} T^*} = \mathcal{H}}$ as $\mathcal{H}$ is a Hilbert space.

Conversely, if $\overline{\operatorname{ran} T^*} = \mathcal{H}$ then $\ker T = \mathcal{H}^\perp = \{0\}$. Thus $\boxed{\ker T = \{0\}}$. □

[13] If a vector is orthogonal to a larger set, it will still be orthogonal to a smaller one.

Analysis Exam

Fall 2015

Problem 5.8.1

Let $\{f_n\}$ be a sequence of analytic functions converging *pointwise* to a continuous function f on a domain D. Show that f is analytic provided each point $z \in D$ has a neighborhood V such that there is a constant M_V such that $|f_n(w)| \leq M_V$ for all n and $w \in V$.

Notes and Comments

Proof. We will prove that the Arzelà–Ascoli Theorem applies to $\{f_n\}$. The given condition on $\{f_n\}$ can be restated as: for each $z \in D$, $\{f_n\}$ is bounded by M_V on a neighborhood V of z. That is, $\boxed{\{f_n\} \text{ is locally bounded}}$. So it suffices to show that $\{f_n\}$ is locally equicontinuous.

Let $z_0 \in D$ and $\varepsilon > 0$. By local boundedness, there is a neighborhood V of z_0 and an associated M_V. Let $r > 0$ be such that $\overline{B_{2r}(z_0)} \subset V \subset D$. By the Cauchy Integral Formula, we have

$$f_n(z) - f_n(w) = \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \left(\frac{f_n(\omega)}{\omega - z} - \frac{f_n(\omega)}{\omega - w} \right) d\omega = \frac{z-w}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{f_n(\omega)}{(\omega - z)(\omega - w)} d\omega$$

for $z, w \in B_{2r}(z_0)$. Further considering $z, w \in B_r(z_0)$, we know $|(\omega - z)(\omega - w)| > r^2$ and so, by the ML-inequality,

$$|f_n(z) - f_n(w)| \leq \frac{|z-w|}{2\pi} \cdot \frac{M_V}{r^2}(2\pi r) = |z-w| \frac{M_V}{r}.$$

Let $\delta = \min\left\{r, \dfrac{r\varepsilon}{M_V}\right\}$. Then, by the above estimate, we have

$$f_n(z) - f_n(w) < \varepsilon$$

for all $z, w \in B_\delta(z_0)$. Thus $\boxed{f_n \text{ is locally equicontinuous}}$ for all n.

By Arzelà–Ascoli, $\{f_n\}$ converges uniformly on compact sets. More precisely, Arzelà–Ascoli guarantees a uniformly convergent subsequence on each compact set (as they are covered by a finite collection of neighborhoods V and so we get a bound by taking the maximum of each given bound). Then, since $f_n \to f$ pointwise, it must be that $f_n \to f$ uniformly on compact sets.[14] Finally, as $f_n \to f$ uniformly on compact sets, f is analytic. □

Problem 5.8.2

Suppose f has an isolated singularity at z_0.

(a) Describe the behavior of $|f(z)|$ near z_0 if z_0 is removable or a pole.

(b) Use the criteria from part (a) to show that if z_0 is an essential singularity for f, then $f(B'_r(z_0))$ is dense in $\mathbb{C}$ for all $r > 0$. Here $B'_r(z_0)$ is the deleted neighborhood $\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$. (Hint: suppose to the contrary that there is a $\omega \in \mathbb{C}$ and $r, \varepsilon > 0$ such that $|f(z) - \omega| \geq \varepsilon$ for all $z \in B'_r(z_0)$. I hope it is clear that you can't evoke the Picard Theorem here.)

[14] Thus we do not need to take subsequences at all.

Notes and Comments

Proof of (a). If z_0 is a removable singularity, then $\lim_{z\to z_0} |f(z)| = L$ for some $L \in \mathbb{R}$.

If z_0 is a pole, then $\lim_{z\to\infty} |f(z)| = \infty$. $\square$

Proof of (b). Following the hint, suppose that $f(B_r'(z_0))$ is not dense in $\mathbb{C}$. Then $\exists \omega \in \mathbb{C}$ and $r, \varepsilon > 0$ such that $\boxed{|f(z) - \omega| \geq \varepsilon}$ $(*)$ for all $z \in B_r'(z_0)$.

Define $g(z) := \frac{1}{f(z)-\omega}$. Then g is holomorphic on $B_r'(z_0)$ and, by $(*)$, $|g(z)| \leq \frac{1}{\varepsilon}$ in this region. Thus g can be extended holomorphically to a function $\overline{g}$ which is holomorphic on $B_r(z_0)$. We consider two cases:

- Suppose that $\overline{g}(z_0) \neq 0$. Then

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{1}{g(z)} + \omega < \infty$$

  and so f has a removable singularity at z_0.

- Suppose that $\overline{g}(z_0) = 0$. Then

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{1}{g(z)} + \omega = \infty$$

  and so f has a pole at z_0.

In either case, the singularity of at z_0 cannot be essential. $\lightning$ Thus the image of f on $B_r'(z_0)$ is dense. $\square$

Problem 5.8.3

If X is a topological space, then $\mathcal{B}(X)$ is the σ-algebra generated by the open sets in X – that is, the Borel sets. If $\mathscr{M}$ and $\mathscr{N}$ are sigma-algebras in X, then $\mathscr{M} \otimes \mathscr{N}$ is the σ-algebra generated by the measurable rectangles $A \times B$ with $A \in \mathscr{M}$ and $B \in \mathscr{N}$. Show that

$$\mathcal{B}(\mathbb{R} \times \mathbb{R}) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$$

where $\mathbb{R}$ is the real line with its usual topology. (Hint: Consider $\mathscr{N} = \{A \ : \ \mathbb{R} \times A \in \mathcal{B}(\mathbb{R} \times \mathbb{R})\}$.)

Notes and Comments

Proof. Let $A \times B \subseteq \mathbb{R}^2$ be an open rectangle. Then $A, B \in \mathcal{B}(\mathbb{R})$ and so $A \times B$ is a measurable rectangle, i.e., $A \times B \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$. Since $\{A \times B \mid A, B \in \mathscr{T}_{\mathbb{R}}\}$ (where $\mathscr{T}_{\mathbb{R}}$ denotes the topology on $\mathbb{R}$) generate the usual topology on $\mathbb{R}^2$ (and thus $\mathcal{B}(\mathbb{R}^2)$), we have $\boxed{\mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})}$.

For the reverse inequality, let $\mathscr{N} = \{B \mid \mathbb{R} \times B \in \mathcal{B}(\mathbb{R}^2)\}$. We claim that $\mathscr{N}$ is a σ-algebra.

Proof. (1) Note that $\mathbb{R} \in \mathscr{N}$ since $\mathbb{R} \times \mathbb{R} \in \mathcal{B}(\mathbb{R}^2)$. Thus $\boxed{\mathscr{N} \neq \varnothing}$.

(2) Let $B \in \mathscr{N}$. We show that $B^c \in \mathscr{N}$. Indeed,

$$\mathbb{R} \times B^c = (\mathbb{R} \times B)^c \in \mathcal{B}(\mathbb{R}^2)$$

since $\mathcal{B}(\mathbb{R}^2)$ is a σ-algebra. Hence $\boxed{B^c \in \mathscr{N}}$.

(3) Let $B_i \in \mathscr{N}$ for $i \in \mathbb{N}$. Then

$$\mathbb{R} \times \left(\bigcup_{i\in\mathbb{N}} B_i \right) = \bigcup_{i\in\mathbb{N}} (\mathbb{R} \times B_i) \in \mathcal{B}(\mathbb{R}^2)$$

and so $\boxed{\bigcup_{i\in\mathbb{N}} B_i \in \mathscr{N}}$.

Thus $\mathscr{N}$ is a σ-algebra. □

Now, certainly we have $\mathscr{T}_{\mathbb{R}} \subset \mathscr{N}$ and so $\mathscr{N} \supset \mathcal{B}(\mathbb{R})$. Hence, for every $B \in \mathcal{B}(\mathbb{R}) \subset \mathscr{N}$, we have $\mathbb{R} \times B \in \mathcal{B}(\mathbb{R}^2)$. That is

$$\boxed{\mathscr{N}' = \{\mathbb{R} \times B \mid B \in \mathcal{B}(\mathbb{R})\} \subset \mathcal{B}(\mathbb{R}^2)}.$$

Similarly, we have

$$\boxed{\mathscr{M}' = \{A \times \mathbb{R} \mid A \in \mathcal{B}(\mathbb{R})\} \subset \mathcal{B}(\mathbb{R}^2)}.$$

Since $\mathcal{B}(\mathbb{R}^2)$ is closed under intersections,

$$\mathcal{B}(\mathbb{R}^2) \supset \mathscr{M}' \cap \mathscr{N}' = \{A \times B \mid A, B \in \mathcal{B}(\mathbb{R})\},$$

the generating set for $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. Hence $\boxed{\mathcal{B}(\mathbb{R}^2) \supseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})}$.

Thus we have $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ as desired. □

Problem 5.8.4

Let X be a set. Prove the vector space $B(X, \mathbb{C})$ of bounded complex–valued maps on X is a Banach space for a norm to be specified.

Notes and Comments

Proof. To show that $B(X, \mathbb{C})$ is a Banach space, we must select a norm and show that it is complete with respect to this norm. Choose the supremum norm, $\|f\| = \sup\{|f(x)| : x \in X\}$. Since f is bounded, $\|f\| < \infty$ and so $\|\cdot\|$ is actually a norm.

Let $\{f_n\} \subset B(X, \mathbb{C})$ be a Cauchy sequence. For each $x \in X$, we know that $\{f_n(x)\} \subset \mathbb{C}$ is a Cauchy sequence and so, by the completeness of $\mathbb{C}$, it must converge. Thus we can define $f : X \to \mathbb{C}$ by $\boxed{f(x) = \lim_{n\to\infty} f_n(x)}$.

We claim that $\{f_n\}$ converges to f.[15] Note that this is certainly true pointwise by the definition of f. So let $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ such that, for all $m, n \geq N$, we have $\|f_n - f_m\| < \varepsilon$. Hence, by the continuity of norms,

$$\|f - f_m\| = \left\| \lim_{n\to\infty} f_n - f_m \right\| = \lim_{n\to\infty} \|f_n - f_m\| < \varepsilon$$

for $m \geq N$. Thus f_n converges to f in norm as well. Hence the sequence converges and $B(X, \mathbb{C})$ is a Banach space as desired. □

[15] If there is any justice in the universe.

Problem 5.8.5

Let E be a separable Banach space.

(a) Recall the definition of the weak topology on E.

(b) Prove that a norm-convergent sequence is weakly convergent. Does the converse hold?

(c) Recall the definition of the weak-$*$ topology on E^* and show that a weakly convergent sequence in E^* is also weak-$*$ convergent.

(d) Prove that a bounded sequence in E^* has a weak-$*$ convergent subsequence.

Notes and Comments

Proof of (a). Let E^* be the topological dual space of E, i.e., the collection of linear functionals $\varphi : E \to \mathbb{F}$ which are continuous with respect to the topology on E induced by the norm. The *weak topology* on E is the coarsest topology such that each functional $\varphi \in E^*$ is still continuous. □

Proof of (b). Let $\{x_n\} \to x$ be norm-convergent. We show that $\{\varphi(x_n)\} \to \varphi(x)$ for all $\varphi \in E^*$, i.e., that $\{x_n\}$ weakly converges to x. Indeed, since φ is continuous with respect to the norm topology on E, this is immediate.

We claim that the converse does not hold. Consider $E = c_0(\mathbb{F})$, the sequences in $\mathbb{F}$ whose limits are 0. We know that E is a Banach space because it is a closed subset of $(\ell^\infty(\mathbb{F}), \|\cdot\|_\infty)$. For the standard basis $\{e_n\}$, we have

$$\|e_n\|_\infty = 1$$

and thus $\boxed{\{e_n\} \text{ does not converge to 0 in norm}}$. However, for any $\varphi \in E^*$, we have

$$\lim_{n\to\infty} \varphi(e_n) = \varphi\left(\lim_{n\to\infty} e_n\right) = \varphi(0) = 0.$$

Thus $\boxed{\{e_n\} \text{ weakly converges to 0}}$. □

Proof of (c). Consider the embedding of E into its double dual E^{**} given by $x \mapsto T_x$ where $T_x(\varphi) = \varphi(x)$. The *weak-$*$ topology* on E^* is the coarsest topology on E^* such that $\{T_x \mid x \in E\}$ is a continuous family of maps.

Assume $\{\varphi_n\}$ is weakly convergent to $\varphi \in E^*$. Then, by definition, for all $T_x \in E^{**}$, we have $T_x(\varphi_n) \to T_x(\varphi)$. That is, $\varphi_n(x) \to \varphi(x)$ in $\mathbb{F}$. As x was arbitrary, we have $\varphi_n(x) \to \varphi(x)$ for all $x \in E$. This is the definition of weak-$*$ convergence, i.e., $\{\varphi_n\}$ is weak-$*$ convergent to φ as desired. □

Proof of (d). Assume $\{\varphi_n\}$ is a bounded sequence in E^*. Let $Q = \{q_1, q_2, \dots\}$ be a countable dense subset of E.[16] By boundedness of the φ_n's, the collection $\{\varphi_n(q)\} \subset \mathbb{F}$ is a bounded set of elements of $\mathbb{F}$ for each $q \in Q$. By Bolzano–Weierstrass, there is a convergent subsequence $\{\varphi_n(q)\}_{n\in\mathbb{N}_q}$ such that $\lim_{n\to\infty} \varphi_{n_i}(q) = \varphi(q)$.

Now we consider $\mathbb{N}_{q_i}$. By an inductive construction, we may assume that $\boxed{\mathbb{N}_{q_1} \supset \mathbb{N}_{q_2} \supset \dots}$ (†). Indeed, start with $\mathbb{N}_{q_i}$ and build $\mathbb{N}_{q_{i+1}}$ by applying Bolzano-Weierstrass to construct the convergent subsequence.

[16] Remember, E is separable. That assumption had to come into play *eventually*.

Define a subsequence of $\{\varphi_n\}$ by taking φ_{n_j} where n_j is the jth element of $\mathbb{N}_{q_j}$. Then, by the nested property (†), we have

$$\lim_{j\to\infty} \varphi_{n_j}(q) = \varphi(q)$$

for all $q \in Q$. Thus we have defined a map $\varphi : Q \to \mathbb{F}$. Since Q is dense in E, φ extends to a bounded linear functional $E \to \mathbb{F}$ such that $\varphi_n(x) \to \varphi(x)$ for all $x \in E$. Thus φ_{n_j} is weak-$*$ convergent to φ. □

Problem 5.8.6

Let S be the map defined on $\ell^2(\mathbb{N})$ by $S(u_0, u_1, \dots) = (0, u_0, u_1, \dots)$.

(a) Prove that S is a bounded linear map between $\ell^2(\mathbb{N})$ and itself and compute $\|S\|$.

(b) Compute the adjoint S^* of S.

(c) Is S a normal operator? Is S an isometry?

Notes and Comments

Proof of (a). It's clear that S is a linear map, so it suffices to prove that $\|S\| < \infty$. For $u = (u_0, u_1, \dots)$, note that

$$\|S(u)\|_2^2 = \sum_{n\geq 0}(S(u))_n^2 = 0 + \sum_{n\geq 0} u_n^2 = \|u\|_2^2 .$$

Hence $\boxed{\|S\| = 1}$ and S is bounded as desired. □

Proof of (b). The adjoint of S is defined by the equation

$$\langle x, S^*(y)\rangle = \langle S(x), y\rangle = 0 \cdot \overline{y_0} + x_0\overline{y_1} + x_1\overline{y_2} + \dots .$$

Define $S^*(u_0, u_1, \dots) = (u_1, u_2, \dots)$. Then S^* is linear and satisfies the above equation. Hence, by uniqueness, S^* actually is the adjoint of S. □

Proof of (c). In part (a), we proved $\|S(u)\|_2 = \|u\|_2$. Hence S is an isometry. However, S is not normal because

$$SS^*(u_0, u_1, \dots) = (0, u_1, \dots) \neq (u_0, u_1, \dots) = S^*S(u_0, u_1, \dots).$$

That is, $SS^* \neq S^*S$. □

Analysis Exam

Summer 2016

Problem 5.9.1

Let (X, M, μ) be a measure space, and suppose that $f_n : X \to \mathbb{R}$ is a measurable function for each $n \geq 1$. Further, suppose that

$$\sup_{n\geq 1}\{f_n\} \in L^1(X, M, \mu).$$

Show that

$$\int_X \limsup f_n d\mu \geq \limsup \int_X f_n d\mu.$$

Notes and Comments

Proof. Let $g = \sup_{n\geq 1}\{f_n\}$. Certainly $f_n \leq g$ for all n. Thus $g - f_n$ is a non-negative sequence of measurable functions. Using the fact that $\liminf(-h_n) = -\limsup(h_n)(*)$ and Fatou's Lemma (†), we have

$$\begin{aligned}
\int_X g\ d\mu - \int_X \limsup f_n\ d\mu &= \int_X (g - \limsup f_n)\ d\mu \\
&\overset{(*)}{=} \int_X \liminf(g - f_n)\ \mu \\
&\overset{(\dagger)}{\leq} \liminf \int_X (g - f_n)\ d\mu \\
&= \int_X g\ d\mu + \liminf \int_X -f_n\ d\mu \overset{(*)}{=} \int_X g\ d\mu - \limsup \int_X f_n\ d\mu
\end{aligned}$$

As $g \in L^1(X, M, \mu)$, we know that $\int_X g\ d\mu$ is finite. Subtracting $\int_X\ d\mu$ from both sides and flipping the inequality, we obtain $\boxed{\int_X \limsup f_n\ d\mu \geq \limsup \int_X f_n\ d\mu}$ as desired. □

Problem 5.9.2

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $\mathbb{N}$ be the set of natural numbers, let $\mathcal{P}(\mathbb{N})$ be the power set of $\mathbb{N}$, and let ν be the counting measure. Consider the product measure space $(\mathbb{N} \times X, \mathcal{P}(\mathbb{N}) \otimes \mathcal{M}, \nu \times \mu)$. (Here $\mathcal{P}(\mathbb{N}) \otimes \mathcal{M}$ is the product σ-algebra.)

(a) **Let $E \subset \mathbb{N} \times X$ and, for $n \in \mathbb{N}$, let $E_n = \{x \in X : (n, x) \in E\}$. Show that $E \in \mathcal{P}(\mathbb{N}) \otimes \mathcal{M}$ if and only if $E_n \in \mathcal{M}$ for every $n \in \mathbb{N}$.**

(b) **Given a function $F : \mathbb{N} \times X \to \mathbb{R}$ and $n \in \mathbb{N}$, define $F_n : X \to \mathbb{R}$ by $F_n(x) := F(n, x)$. Show that F is $(\mathcal{P}(\mathbb{N}) \otimes \mathcal{M})$-measurable if and only if each function F_n, $n \in \mathbb{N}$, is $\mathcal{M}$-measurable.**

(c) **Interpret Tonelli's Theorem in this setting. (Recall that Tonelli's Theorem is the part of the Fubini–Tonelli Theorem concerning nonnegative functions.) More precisely, indicate what familiar result is equivalent to the equality of the iterated integrals.**

Notes and Comments

Proof of (a). ($\Leftarrow$): Recall that the product σ-algebra is generated by measurable rectangles. So, if each $E_n \in \mathcal{M}$ then

$$E = \bigcup_{n \in \mathbb{N}} \{n\} \times E_n$$

is a countable union of measurable sets. Thus $E \in \mathcal{P}(\mathbb{N}) \otimes \mathcal{M}$.

($\Rightarrow$): Assume $E \in \mathcal{P}(\mathbb{N}) \otimes \mathcal{M}$. Then E is the union of measurable rectangles. Since ν is the counting measure and $\mathbb{N}$ is countable, we may assume that each measurable rectangle consists of the n-slice $\{n\} \times E_n$.[17] That is E_n must be $\mathcal{M}$-measurable for all n. □

Proof of (b). Note that, for any measurable $U \subset \mathbb{R}$, we have

$$F^{-1}(U) = \bigcup_{n=1}^{\infty} \{n\} \times F_n^{-1}(U).$$

($\Leftarrow$): Assume that F_n is $\mathcal{M}$-measurable for all n. Then $F_n^{-1}(U)$ is $\mathcal{M}$-measurable. So $F^{-1}(U)$ is union of measurable rectangles and hence $(\mathcal{P}(\mathbb{N}) \otimes \mathcal{M})$-measurable. Thus F is measurable as desired.

($\Rightarrow$): Assume F is $(\mathcal{P}(\mathbb{N}) \otimes \mathcal{M})$-measurable. Then $F^{-1}(U)$ is $(\mathcal{P}(\mathbb{N}) \otimes \mathcal{M})$-measurable. By part (a), $F_n^{-1}(U)$ is $\mathcal{M}$-measurable. That is, each F_n is $\mathcal{M}$-measurable. □

Proof of (c). Let $f \in L^+(\mathbb{N} \times X)$. Then Tonelli's Theorem lets us conclude that

$$\int_{\mathbb{N}} \int_X f(n,x)\ d\mu(x) d\nu(n) = \int_X \int_{\mathbb{N}} f(n,x)\ d\nu(n) d\mu(x).$$

That is,

$$\sum_{n=1}^{\infty} \int_X f(n,x)\ d\mu(x) = \int_X \sum_{n=1}^{\infty} f(n,x)\ d\mu(x).$$

That is, for absolutely convergent series of integrals, we can interchange the integral and the sum. □

Problem 5.9.3

For each of the following, explain why there *cannot* be a function $f : \mathbb{C} \to \mathbb{C}$ with the stated properties.

(a) f **is an entire function and** $\int_C f(z)dz = 5$**, where** C **is the positively oriented circle** $|z| = 1$**.**

(b) f **is entire,** $f(yi) = yi$ **for** $0 \le y \le 1$ **and** $f(7 + 2i) = 2i$**.**

(c) f **is entire and** $|f(x + yi)| = e^{-(x^4+y^4)}$ **for all** $x + yi \in \mathbb{C}$**.**

(d) f **is entire,** f **has a zero of order 5 at the origin and** $\int_C f(\frac{1}{z})dz = 2\pi i$**, where** C **is the positively-oriented circle** $|z| = 1$**.**

Notes and Comments

[17] Each rectangle R_i splits into a union of $\{n\} \times R_{i,n}$. Then E_n is the union of $R_{i,n}$ for $i \in \mathbb{N}$. Since $\mathcal{M}$ is a σ-algebra, E_n is measurable.

Proof of (a). Since f is entire, we have $f \in H(\mathbb{C})$ by definition. As C is a closed curve and $\mathbb{C}$ is simply-connected, we have $\int_C f(z)dz = 0$ by Cauchy's Theorem. $\lightning$ □

Proof of (b). Consider the identity map Id. Then $f(z) = \mathrm{Id}(z)$ on $S = \{yi \mid 0 \le y \le 1\}$. As $i, 0$ are accumulation points of S in $\mathbb{C}$, we have $f = \mathrm{Id}$ by the Identity Theorem[18]. Thus $f(7+2i) = 7+2i \ne 2i$. $\lightning$ □

Proof of (c). If $|f(x+yi)| = e^{-(x^4+y^4)}$, then, since $x^4 + y^4$ is non-negative, we have $|f(z)| \in (0,1]$ for all $z \in \mathbb{C}$. By Liouville's Theorem, all bounded entire functions are constant. However, f is not constant. $\lightning$ □

Proof of (d). Since f is entire, it has a power series expansion $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ about the origin. Moreover, as f has a zero of order 5 at the origin, we know $a_0 = a_1 = \cdots = a_4 = 0$. Thus $f(\frac{1}{z}) = \displaystyle\sum_{n=5}^{\infty} a_n z^{-n}$ is a valid Laurent expansion on $\mathbb{C} \setminus \{0\}$.

Finally, the hypothesis that $\int_C f(\frac{1}{z})\, dz = 2\pi i$ says exactly that the residue about its singularity at zero of $f(\frac{1}{z})$ is 1. That is, $f(\frac{1}{z})$ only has a singularity at $z = 0$ (since f is entire) and C is a closed curve around 0 with winding number 1, so the equality holds by the Residue Theorem.

Since the residue, by definition, is the coefficient of z^{-1} in the Laurent expansion about 0, we have $1 = \mathrm{Res}(f(\frac{1}{z}); 0) = a_1 = 0$. $\lightning$ □

Problem 5.9.4

Let X and Y be Banach spaces. Suppose that $T : X \to Y$ is a linear map such that if (x_n) is a sequence converging weakly to 0 in X, then $(T(x_n))$ converges weakly to zero in Y. Show that T is bounded.

Notes and Comments

Proof. We use the Closed Graph Theorem. Assume that $(s_n) \to s$ and $(T(s_n)) \to y$ in norm. We show that $T(s) = y$.

Since $(s_n) \to s$ in norm, we have $(s_n - s) \to 0$ in norm. Thus $(s_n - s) \to 0$ weakly. By the hypothesis on T, we have $(T(s_n) - s) \to 0$ weakly. By linearity of T, we thus have $\boxed{(T(s_n)) \to T(s) \text{ weakly}}$.

On the other hand, since $(T(s_n)) \to y$ in norm, it must be that $\boxed{(T(s_n)) \to y \text{ weakly}}$. As the weak topology is Hausdorff, weak limits are unique. Thus $\boxed{T(s) = y}$. By the Closed Graph Theorem, T is bounded. □

Problem 5.9.5

Let $A = C([0,1])$ with the uniform norm. Let $K \in C([0,1] \times [0,1])$. For each $f \in A$ and $x \in [0,1]$, define

$$T(f)(x) = \int_0^1 K(x,y) f(y)\, dy.$$

(a) Show that $T(f) \in A$.

(b) Show that $T \in \mathcal{L}(A)$.

[18] If the agreement set of two holomorphic functions $D \to \mathbb{C}$ has an accumulation point in the domain D, then the two functions are identical.

(c) Let $B = \{f \in A : \|f\|_\infty \leq 1\}$. Show that $\overline{T(B)}$ is compact in A.

Notes and Comments

Proof of (a). As K, f are continuous maps defined on compact sets, they obtain maximums $\|K\|_\infty$ and $\|f\|_\infty$ respectively. Then

$$|T(f)(x)| \leq \int_0^1 |K(x,y)f(y)| \; dy \leq \int_0^1 \|K\|_\infty \|f\|_\infty \; dy = \|K\|_\infty \|f\|_\infty .$$

Define $h(y) = \|K\|_\infty \|f\|_\infty$ for $y \in [0,1]$. Then $h \in L^1([0,1])$ since $[0,1]$ is a finite measure space.

If $x \neq 1$, let $g_n(y) = K(x + \frac{1}{n}, y)f(y)$ for large n.[19] Since K is continuous, $\lim\limits_{n\to\infty} g_n = K(x,\cdot)f(\cdot)$. Moreover, $|g_n| \leq h$ for all n. Hence, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^1 K\left(x + \frac{1}{n}, y\right) f(y) \; dy = \lim_{n\to\infty} \int_0^1 g_n(y) \; dy = \int_0^1 K(x,y)f(y) \; dy = T(f)(x).$$

Thus $T(f)$ is continuous at x.

If $x = 1$, take $g_n(y) = K(x - \frac{1}{n}, y)f(y)$. Then the same story plays out.

So $T(f) \in A$ as desired. □

Proof of (b). Since integration is linear, T is a linear map. Now we show that T is bounded.

Indeed, using the bound established in part (a),

$$\|T(f)\|_\infty = \sup_{x\in[0,1]} |T(f)(x)| \leq \sup_{x\in[0,1]} \|K\|_\infty \|f\|_\infty = \|K\|_\infty \|f\|_\infty .$$

As f is arbitrary, $\|T\|_\infty \leq \|K\|_\infty$ and so T is bounded. □

Proof of (c). To show that $\overline{T(B)}$ is compact, it suffices to show that any sequence $\{T(f_n)\}_{n=1}^\infty$ in $T(B)$ has a convergent subsequence (which will necessarily be in $\overline{T(B)}$). Unsurprisingly, we will use the Arzelà–Ascoli Theorem.

By definition of B, $\|f_n\|_\infty \leq 1$. Thus, by the bound in part (b), we have $\|T(f_n)\|_\infty \leq \|K\|_\infty$ for all n. That is, $\boxed{\{T(f_n)\}_{n=1}^\infty \text{ is uniformly bounded}}$.

To show that $\{T(f_n)\}_{n=1}^\infty$ is equicontinuous, let $\varepsilon > 0$ and $x_1, x_2 \in [0,1]$. Then

$$|T(f_n)(x_1) - T(f_n)(x_2)| \leq \int_0^1 |K(x_1,y) - K(x_2,y)| \, |f_n(y)| \; dy \leq \int_0^1 |K(x_1,y) - K(x_2,y)| \, \|f_n\|_\infty \; dy \leq \int_0^1 |K(x_1,$$

By the Heine–Cantor Theorem, K is uniformly continuous. Hence $\exists \delta > 0$ such that $|x_1 - x_2| < \delta \implies |K(x_1,y) - K(x_2,y)| < \varepsilon$.[20] Therefore, for $|x_1 - x_2| < \delta$, we have

$$|T(f_n)(x_1) - T(f_n)(x_2)| \leq \int_0^1 |K(x_1,y) - K(x_2,y)| \; dy < \int_0^1 \varepsilon \; dy = \varepsilon.$$

That is, $T(f_n)$ is equicontinuous for all n.

By Arzelà–Ascoli, $\{T(f_n)\}_{n=1}^\infty$ has a convergent subsequence. Thus $\overline{T(B)}$ is compact as desired. □

[19] Specifically such that $x + \frac{1}{n} < 1$.

[20] Since $[0,1]^2$ is two-dimensional, we should have a different distance function here. However, once we have that, we can restrict to the one dimension that we care about.

Problem 5.9.6

Let X be a compact metric space. Suppose that $f_n : X \to \mathbb{R}$ is continuous for each $n \geq 1$, and that for each $x \in X$, $f_{n+1}(x) \leq f_n(x)$. Show that if $f_n \to 0$ pointwise on X, then $f_n \to 0$ uniformly on X.

Notes and Comments

Proof. Let $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ such that for $n \geq N$, $|f_n(x)| < \varepsilon$ for all $x \in X$. The idea is to produce an open cover of X and exploit the fact that X is compact to produce a finite subcover. Consider $U_n = f_n^{-1}((-\varepsilon, \varepsilon))$ for $n \in \mathbb{N}$.[21]
Claim 1: $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of X.

Proof. Since f_n is continuous, U_n is open for each n. Note that if $x \in X$, then $f_n(x) \to 0$ by hypothesis. Therefore $\exists n$ such that $|f_n(x)| < \varepsilon$. Thus $x \in U_n$. Hence $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of X. $\square$

By compactness of X, there exists $N \in \mathbb{N}$ such that $\{U_1, U_2, \ldots, U_N\}$ is an open cover of X.
Claim 2: $U_n \subset U_{n+1}$ for any $n \in N$.

Proof. Suppose $x \in U_n$. Then $-\varepsilon < f_n(x) < \varepsilon$, so $f_{n+1}(x) \leq f_n(x) < \varepsilon$. Also $f_k(x) \searrow 0$ and so $f_k(x) \geq 0$ for all k. In particular, $-\varepsilon < 0 \leq f_n(x)$. Thus $x \in U_{n+1}$. $\square$

By Claim 2, the open cover $\{U_1, U_2, \ldots, U_N\}$ forms an ascending chain $U_1 \subset U_2 \subset \cdots \subset U_N$. This means that U_N covers X. Hence $X = U_N = U_{N+1} = U_{N+2} = \ldots$. Thus $x \in U_n$ for all $n \geq N$. By definition of U_n, this means $|f_n(x)| < \varepsilon$. Therefore, $f_n \to 0$ uniformly as desired. $\square$

[21] A hypothetical desperate graduate student solving this problem during a qual would argue that $\{U_n\}_{n\in\mathbb{N}}$ is a good candidate for an open cover. Since there are not many ways to produce a collection of open sets starting with $\{f_n\}_{\mathbb{N}}$, we will go with this.

Analysis Exam

Fall 2016

Problem 5.10.1

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $g \in L^+(X, \mathcal{M})$, i.e., $g : X \to [0, \infty]$ is a measurable function. Define $\nu : \mathcal{M} \to [0, \infty]$ by

$$\nu(E) = \int_E g \, d\mu.$$

(a) Show that ν is a measure on $(X, \mathcal{M})$.

(b) Show that for any nonnegative measurable function $f : X \to [0, \infty]$, we have

$$\int_X f \, d\nu = \int_X fg \, d\mu.$$

(c) Suppose that $g \in L^2(X, \mathcal{M}, \mu)$. Show that

$$L^2(X, \mathcal{M}, \mu) \subset L^1(X, \mathcal{M}, \nu).$$

Notes and Comments

Proof of (a). First note that $\boxed{\nu(\varnothing) = \int_\varnothing h \, d\mu = 0}$. Now let $\{E_n\}_{n=1}^\infty$ be a disjoint collection of measurable sets. Since $g \in L^+(X, \mathcal{M})$, there is a sequence of simple functions φ_i such that $\varphi_i \nearrow g$. Furthermore, $E \mapsto \int_E \varphi \, d\mu$ is a measure for any simple function φ. Hence we have $\boxed{\int_{\bigcup_{n=1}^\infty E_n} \varphi_i \, d\mu = \sum_{n=1}^\infty \int_{E_n} \varphi_i \, d\mu}(*)$. By the Monotone Convergence Theorem (MCT) and the fact that our sum converges absolutely (†), we have

$$\begin{aligned}
\nu\left(\bigcup_{n=1}^\infty E_n\right) = \int_{\bigcup_{n=1}^\infty E_n} g \, d\mu &\overset{(MCT)}{=} \lim_{i\to\infty} \int_{\bigcup_{n=1}^\infty E_n} \varphi_i \, d\mu \\
&\overset{(*)}{=} \lim_{i\to\infty} \sum_{n=1}^\infty \int_{E_n} \varphi_i \, d\mu \\
&\overset{(\dagger)}{=} \sum_{n=1}^\infty \lim_{i\to\infty} \int_{E_n} \varphi_i \, d\mu \\
&\overset{(MCT)}{=} \sum_{n=1}^\infty \int_{E_n} g \, d\mu = \sum_{n=1}^\infty \nu(E_n).
\end{aligned}$$

Thus ν is a measure. □

Proof of (b). First consider a simple function $\varphi = \sum_i z_i \chi_{E_i}$. Then

$$\int_X \varphi \, d\nu = \sum_i z_i \int_X \chi_{E_i} \, d\nu = \sum_i z_i \nu(E_i) = \sum_i z_i \int_{E_i} g \, d\mu = \int_X \varphi g \, d\mu.$$

As $f \in L^+(X, \mathcal{M})$, there is a sequence of simple functions φ_n such that $\varphi_n \nearrow f$. Hence, by repeated applications of the Monotone Convergence Theorem,

$$\int_X f \, d\nu = \lim_{n\to\infty} \int_X \varphi_n \, d\nu = \lim_{n\to\infty} \int_X \varphi_n g \, d\mu = \int_X fg \, d\mu.$$

Thus we have the desired equality. □

Proof of (c). Let $f \in L^2(X, \mathcal{M}, \mu)$ and $\|\cdot\|_p$ denote the p-norm on $L^p(X, \mathcal{M}, \mu)$. Then we have

$$\int_X |f| \; d\nu \overset{(b)}{=} \int_X |f| \, g \, d\mu = \int_X |fg| \; d\mu = \|fg\|_1 \le \|f\|_2 \, \|g\|_2 < \infty$$

by Hölder's Inequality ($p = q = 2$ are conjugate) and the fact that $\|f\|_2 \, , \|g\|_2 < \infty$ by assumption. Thus $f \in L^1(X, \mathcal{M}, \nu)$ by definition. □

Problem 5.10.2

Let m^* be the Lebesgue outer measure on $\mathbb{R}$. Let E be a subset of $\mathbb{R}$ with $m^*(E) < \infty$.

(a) Show that there exists a Borel set B such that $E \subset B$ and $m^*(B) = m^*(E)$.

(b) Let B be as in part (a). Show that E is Lebesgue measurable if and only if $m^*(B - E) = 0$. (You may use the facts that Borel sets are Lebesgue measurable and that the Lebesgue measure is complete.)

Notes and Comments

Proof of (a). For all $n \in \mathbb{N}$, there is a covering of E by $\{I_{n,k}\}_k$ such that $m^*\left(\bigcup_k I_{n,k}\right) \le m^*(E) + \dfrac{1}{n}$ by the definition of m^*. As each interval $I_{n,k}$ is a Borel set (being a closed interval) and Borel sets for a σ-algebra, we know that $B_n = \bigcup_k I_{n,k}$ is a Borel set for each n. Moreover, $\boxed{E \subset B_n}(*)$ for all n.

Consider $\boxed{B = \bigcap_n B_n}$. Since σ-algebras are closed under countable intersections, we know that B is Borel. Also, $\boxed{B \subset B_n}(\dagger)$ for all n. Thus we have

$$m^*(E) \overset{(*)}{\le} m^*(B) \overset{(\dagger)}{\le} m^*(B_n) \le m^*(E) + \frac{1}{n}$$

for all n. Taking the limit with respect to n, we obtain $m^*(E) = m^*(B)$ as desired. □

Proof of (b). ($\Rightarrow$) Assume E is Lebesgue measurable. Then, since $E \subset M$ and $m^*(E) = m^*(B)$,

$$m^*(B) = m^*(B \cap E) + m^*(B \cap E^c) = m^*(E) + m^*(B - E) = m^*(B) + m^*(B \setminus E).$$

As $m^*(E) = m^*(B) < \infty$ (by the original hypotheses), we obtain $m^*(B \setminus E) = 0$ as desired.

($\Leftarrow$) Assume $m^*(B - E) = 0$. Since the Lebesgue measure is complete, $B - E$ is Lebesgue measurable. As Lebesgue measurable sets form a σ-algebra,

$$E = B - (B - E) = B \cap (B - E)^c$$

is Lebesgue measurable. □

Problem 5.10.3

(a) State the Cauchy Integral Formulas for analytic functions f and their derivatives $f^{(n)}(z)$.

(b) Use the Cauchy Integral Formulas to derive the Cauchy estimates which give bounds for f and its derivatives at a point z_0 in terms of the maximum of f on a circle $|z - z_0| = R$. Be sure to state any hypothesis.

(c) State and prove Liouville's Theorem.

Notes and Comments

Theorem 5.10.1 (CAUCHY INTEGRAL FORMULAS): *Assume f is analytic on a domain U containing the closed disk $\overline{D_r(z_0)}$ for some $z_0 \in U$ and $r > 0$. Let γ parametrize the circle $\partial D_r(z_0)$ with a counterclockwise orientation. Then, for all z in the open disk $D_r(z_0)$ and all $n \geq 0$,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Proof of (b). Under the assumptions in (a), suppose also that $M = \max\limits_{z \in \partial D_r(z_0)} |f(z)|$. Then, by the ML-inequality $(*)$, we have

$$\begin{aligned} |f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| &= \frac{n!}{2\pi} \left| \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| \\ &\leq \frac{n!}{2\pi} \int_\gamma \left| \frac{f(z)}{(z - z_0)^{n+1}} \right| dz \\ &\overset{(*)}{\leq} \left(\frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \right) \cdot 2\pi r = \frac{n!M}{r^n}. \end{aligned}$$

That is, Cauchy Estimates give $\boxed{|f^{(n)}(z_0)| \leq \dfrac{n!M}{r^n}}$. ☐

Theorem 5.10.2 (LIOUVILLE'S THEOREM): *Every bounded entire function is constant.*

Proof of (c). Assume f is an entire function which is bounded by M. Let $z \in \mathbb{C}$ and take $r > 0$. From part (b), we have

$$|f'(z)| \leq \frac{M}{r}.$$

Taking a limit as $r \to \infty$, we see that $f'(z) = 0$. As z was arbitrary, it must be that f is constant. ☐

Problem 5.10.4

Suppose that $T : X \to Y$ is a linear map between Banach spaces. Show that T is bounded if and only if T is continuous at 0.

Notes and Comments

Proof. ($\Rightarrow$) Assume that $T : X \to Y$ is bounded. Then there exists some $M > 0$ such that $||T(x)|| \le M||x||$ for all $x \in X$.

Let $\varepsilon > 0$. Since T is linear, we know $T(0) = 0$. Thus $\|T(x) - T(0)\| = \|T(x)\| \le M\|x\|$. Take $\delta = \frac{\varepsilon}{M} > 0$. Then whenever $\|x - 0\| < \delta$, we have $\|T(x) - T(0)\| \le M\|x\| < \varepsilon$. Hence T is continuous at 0.

($\Leftarrow$) Assume that T is continuous at 0. Let $\varepsilon = 1$. Then there exists δ for which $||x|| < \delta \implies ||T(x)|| < 1$.

Let $x \in X$ with $x \neq 0$. Then

$$\left\| \frac{\delta}{2\|x\|} x \right\| = \frac{\delta}{2} < \delta.$$

Hence $\left\| T\left(\frac{\delta}{2\|x\|} x \right) \right\| < 1$. By linearity, we see

$$T(x) < \frac{2}{\delta}\|x\|.$$

Thus $M = \dfrac{2}{\delta}$ shows that T is bounded. □

Problem 5.10.5

Let $T : H \to H$ and $S : H \to H$ be functions (not necessarily linear) from a Hilbert space H to itself. Suppose that for each $x, y \in H$ we have

$$(T(x) \mid y) = (x \mid S(y)).$$

Show that T and S are bounded linear maps with $S = T^*$.

Notes and Comments

Proof. First we show that T is linear. Indeed, for $x, y \in H$, we have

$$(T(x+y) \mid z) = (x + y \mid S(z)) = (x \mid S(z)) + (y \mid S(z)) = (T(x) \mid z) + (T(y) \mid z) = (T(x) + T(y) \mid z).$$

Thus $T(x+y) - T(x) - T(y)$ is orthogonal to H. Hence $T(x+y) = T(x) + T(y)$. Similarly, $T(\alpha x) = \alpha T(x)$ and so $\boxed{T \text{ is linear}}$.

The proof that T is bounded follows from problem 4(b) on the Summer 2012 exam (5.1.4). That is, we apply the Closed Graph Theorem to conclude that T is bounded. Hence, as S is a map that satisfies the adjoint property for a bounded linear map T, S is the adjoint of T. (So S is also a bounded linear map.) □

Problem 5.10.6

We say $f : [0, 1] \to \mathbb{R}$ is α–Hölder if

$$h_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

For $M > 0$ and $\alpha \in (0, 1]$, let

$$H_{\alpha,M} := \{f \in C([0, 1]) : h_\alpha(f) \le M \text{ and } ||f||_\infty \le M\}.$$

Show that $H_{\alpha,M}$ is compact in $C([0,1])$.

Notes and Comments

Proof. To show that $H_{\alpha,M}$ is compact, it suffices to show that every sequence contains a convergent subsequence. We want to apply the Arzelà–Ascoli Theorem, so we need to show that any sequence in $H_{\alpha,M}$ is uniformly bounded and equicontinuous.

Let $\{f_n\}$ be a sequence in $H_{\alpha,M}$. Uniform boundedness follows from $||f_n||_\infty \leq M$ so it remains to show equicontinuity. Let $\varepsilon > 0$ and take $\delta = \sqrt[\alpha]{\frac{\varepsilon}{M}}$.

Then, for all $x, y \in [0,1]$ with $|x-y| < \delta$ and any $n \in \mathbb{N}$, we have

$$|f_n(x) - f_n(y)| \leq h_\alpha(f_n) \left|x-y\right|^\alpha \leq M \left|x-y\right|^\alpha < M\delta^\alpha < \varepsilon.$$

Hence we satisfy the hypotheses of Arzelà–Ascoli and so $\{f_n\}$ contains a convergent subsequence, implying the compactness of $H_{\alpha,M}$. □

Analysis Exam

Summer 2017

Problem 5.11.1

Let $p \geq 1$ be an integer and f a holomorphic function on $D_1(0)$ such that

(i) $|f(z)| \leq |z|^p$ **for all z with $|z| < 1$;**

(ii) f **has a zero of order $\geq p$ at 0.**

Assume further the existence of $a \neq 0$ such that $f(a) = a^p$. What can be said of f?

Notes and Comments

Proof. It follows from (ii) that $\lim_{z\to 0} f(z)/z^p$ exists and is some finite number $c \in \mathbb{C}$, and that the function

$$g(z) = \begin{cases} \frac{f(z)}{z^p} & z \neq 0 \\ c & z = 0 \end{cases}$$

is holomorphic on $D_1(0)$. We know from (i) that $|g(z)| = \dfrac{|f(z)|}{|z|^p} \leq 1$ for all $z \neq 0$. By continuity of g, we thus have $|g(0)| \leq 1$. Furthermore, since $f(a) = a^p$ for some $a \neq 0$, we have $\boxed{g(a) = 1}$. Thus we see that 1 is the maximum value of g, and this value is attained at a. As the domain $D_1(0)$ is open, g must be constant by the Maximum-Modulus Principle. Hence $g(z) = c$ for all z. Therefore $f(z) = cz^p$, where $c = \lim_{z\to 0} \dfrac{f(z)}{z^p}$. $\square$

Problem 5.11.2

Let $(X, \mathfrak{M}, \mu)$ be a measured space, $(Y, \mathfrak{N})$ a measurable space, and $f\colon X \to Y$ a measurable function. Define for $S \in \mathfrak{N}$: $\nu(S) = \mu(f^{-1}(S))$.

(a) Verify that ν is a measure on $(Y, \mathfrak{N})$.

(b) Let $u\colon Y \to \mathbb{R}$ be a measurable function. Prove that

$$\int_Y u\, d\nu = \int_X u \circ f\, d\mu.$$

Notes and Comments

Proof of (a). See the solution to 2(a) from Summer 2014 (5.5.2). $\square$

Proof of (b). In the case that u is non-negative, see the solution to 2(b) from Summer 2014 (5.5.2). For the general case, define $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Then $u^+, u^- \geq 0$ are non-negative measurable functions, and $u = u^+ - u^-$, so

$$\int_Y u\, d\nu = \int_Y u^+\, d\nu - \int_Y u^-\, d\nu = \int_X u^+ \circ f\, d\mu - \int_X u^- \circ f\, d\mu = \int_X u \circ f\, d\mu.$$

$\square$

Problem 5.11.3

Fun with functions.

(a) **Give an example of a sequence $\{f_n\}_{n\geq 1}$ of measurable functions such that $f_n \xrightarrow{L_1} f$ but f_n does not converge to f almost everywhere.**

(b,c) **For parts (b) and (c), let $\{f_n\}_{n\geq 1}$ and f be functions on a measured space X such that for all n,**

$$||f_n - f||_1 \leq 3^{-n}.$$

(b) **Let $E_n = \{x \in X : |f_n(x) - f(x)| \geq \frac{1}{2^n}\}$ and $G_k = \bigcup_{n\geq k} E_n$. Prove that**

$$\mu(G_k) \leq \frac{2^k}{3^{k-1}}.$$

(c) **Deduce that $f_n \to f$ almost everywhere.**

Notes and Comments

Proof of (a). For any positive integer n, we can uniquely write $n = j+2^k$ for $0 \leq j < 2^k$ and $2^{k-1} \leq n \leq 2^k$. Let $F_n = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$ and define $f_n = \chi_{F_n}$ to be the characteristic function on F_n. Then $f_n \xrightarrow{L_1} f = 0$, but does not converge everywhere.[22] □

Proof of (b). Note that $\mu(G_k) = \mu\left(\bigcup_{n\geq k} E_n\right) \leq \sum_{n\geq k} \mu(E_n)$. Keeping this in mind, let us first find $\mu(E_n)$. We can write

$$||f_n - f||_1 = \int_X |f_n - f| d\mu = \int_{E_n} |f_n - f| d\mu + \int_{E_n^c} |f_n - f| d\mu \leq 3^{-n}.$$

However, $\displaystyle\int_{E_n} |f_n - f| d\mu \geq \frac{1}{2^n}\mu(E_n)$. Rewriting the above, we have

$$\frac{1}{2^n}\mu(E_n) \leq 3^{-n} - \int_{E_n^c} |f_n - f| d\mu \leq 3^{-n}.$$

Thus, $\mu(E_n) \leq \frac{2^n}{3^n}$. Then

$$\mu(G_k) \leq \sum_{n\geq k} \mu(E_n) \leq \sum_{n\geq k} \frac{2^n}{3^n} \leq \sum_{n=1}^{\infty} \frac{2^k}{3^k}\left(\frac{2}{3}\right)^n = \frac{2^k}{3^{k-1}}$$

as desired. □

[22] Every point takes on value 1 infinitely often.

Proof of (c). Let $G = \bigcap_{k=1}^{\infty} G_k$. Then G is the set of all points $x \in X$ such that $|f_n - f| \geq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. We will show that $f_n \to f$ pointwise outside of G and that $\mu(G) = 0$.

Let $x \notin G$ and let $\varepsilon > 0$ be given. Then there exists some N_ε such that $\frac{1}{2^{N_\varepsilon}} < \varepsilon$. Furthermore, as $x \notin G$, we know $x \notin G_{N_x}$ for some $N_x \in \mathbb{N}$. Hence $x \notin G_n$ for all $n \geq N_x$. That is, there exists $N_x \in \mathbb{N}$ such that $|f_n(x) - f(x)| \leq \frac{1}{2^n}$ for all $n \geq N_x$.

Let $N = \max(N_\varepsilon, N_x)$. Then

$$|f_n(x) - f(x)| \leq \frac{1}{2^n} < \varepsilon$$

for all $n \geq N$. Thus $f_n(x) \to f(x)$ for all $x \notin G$.

Finally, since $G_{k+1} \subset G_k$ for all k, continuity of measures from above says that

$$\mu(G) = \lim_{k\to\infty} \mu(G_k) \leq \lim_{k\to\infty} \frac{2^k}{3^{k-1}} = 0.$$

Thus $f_n \to f$ except on a set of measure zero, and so $f_n \to f$ almost everywhere. □

Problem 5.11.4

Let H be a Hilbert space and T a normal operator on H.

(a) Show that, for all $h \in H$, we have $\|Th\| = \|T^*h\|$.

(b) Show that if v is an eigenvector for T, then v is an eigenvector for T^*.

(c) Show that if v and w are eigenvectors for T with eigenvalues λ and μ, respectively, then $v \perp w$ if $\lambda \neq \mu$.

Notes and Comments

Proof of (a). Recall the definition of normal operator: $T^*T = TT^*$. Then using the definition of the adjoint repeatedly, we have

$$\|Th\|^2 = \langle Th, Th\rangle = \langle h, T^*Th\rangle = \langle h, TT^*h\rangle = \langle T^*h, T^*h\rangle = \|T^*h\|^2.$$

As norms are non-negative, we have $\|Th\| = \|T^*h\|$. □

Proof of (b). Let λ be the eigenvalue associated to v. Then $T - \lambda$ is normal.[23] By part (a), we have

$$\|(T-\lambda)v\| = \|(T^* - \lambda^*)v\|,$$

where λ^* is the complex conjugate of λ. Thus

$$\|(T^* - \lambda^*)v\| = \|(T-\lambda)v\| = \|\lambda v - \lambda v\| = 0,$$

and so $T^*v - \lambda^*v = 0$. Therefore, v is an eigenvector for T with eigenvalue λ^*. □

Proof of (c). We have $Tv = \lambda v$ and, by part (b), $T^*w = \mu^*w$. So

$$\lambda\langle v, w\rangle = \langle \lambda v, w\rangle = \langle Tv, w\rangle = \langle v, T^*w\rangle = \langle v, \mu^*w\rangle = \mu\langle v, w\rangle,$$

since the inner product is Hermitian. Thus $(\lambda - \mu)\langle v, w\rangle = 0$. Since $\lambda \neq \mu$, we have $\langle v, w\rangle = 0$. □

[23] It's a quick check.

Problem 5.11.5

Let X be a normed vector space such that X^* is separable. Show that X is separable. (I suggest letting $\{\phi_n\}$ be a countable dense subset of X^* and choose $x_n \in X$ such that $\|x_n\| = 1$ and $\phi_n(x_n) \geq \frac{1}{2}\|\phi_n\|$. Consider $\operatorname{span}(\{x_n\})$.)

Notes and Comments

Proof. Let $D = \{\phi_n\}_{n=1}^{\infty}$ be a countable dense subset of X^*. By definition of the operator norm, $\|\phi_n\| = \sup_{\|x\|=1} |\phi_n(x)|$. So $\exists x_n \in X$ such that $|\phi_n(x_n)| \geq \frac{1}{2}\|\phi_n\|$. If this inequality doesn't hold without absolute values, replace x_n by $-x_n$. Define $S = \{x_n\}_{n=1}^{\infty}$.

Let $Y = \operatorname{span}_{\mathbb{Q}} S$. We claim that $Y \subset X$ is the desired set. Since S is countable and $\mathbb{Q}$ is countable, Y is also countable. So it suffices to show that $\overline{Y} = X$.

As X is a normed space and Y is a linear subspace of X, $\overline{Y} \subseteq X$.[24]

Now suppose that $x \in X \setminus \overline{Y}$. Then, by Hahn–Banach, there is some $\phi \in X^*$ with $\phi \neq 0$ (specifically $\phi(x) \neq 0$) and $\phi|_{\overline{Y}} = 0$. By the density of D in X^*, there is some $\phi_k \in D$ such that $\|\phi - \phi_k\| < \varepsilon$ for any choice of ε.

Note that $\phi(x_k) = 0$ since $x_k \in Y$. However, by construction,

$$\frac{1}{2} \leq |\phi_k(x_k)| = |\phi_k(x_k) - \phi(x_k)| \leq \|\phi_k - \phi\| \|x_k\| < \varepsilon \cdot 1 = \varepsilon.$$

Choose $\varepsilon \leq \frac{1}{2}$ to obtain a contradiction. ↯ Thus no such x can exist and so $\overline{Y} = X$ as desired. Hence Y is a countable dense subset of X, i.e., X is separable. □

Problem 5.11.6

Let $\mathcal{F} \subset C([0,1])$ be the collection of continuous functions such that $f'(x)$ exists and satisfies $|f'(x)| \leq 2$ for all $x \in (0,1)$. Let $\mathcal{F}_0 = \{f \in \mathcal{F} : |f(0)| \leq 3\}$.

(a) Show that $\mathcal{F}$ is equicontinuous on $[0,1]$.

(b) Explain why the closure of $\mathcal{F}$ is not compact, while the closure of $\mathcal{F}_0$ is.

Notes and Comments

Proof of (a). Since $|f'(x)|$ is bounded on $(0,1)$ by 2, $|f'(x)|$ is bounded on $[0,1]$ by some constant $M \geq 2$ where $M = \max\{|f'(0)|, |f'(1)|, 2\}$.

Let $\varepsilon > 0$. Fix $x_0 \in [0,1]$ and $f \in \mathcal{F}$. Since $|f'(x_0)| \leq M$, there is some $\delta' > 0$ such that

$$|x - x_0| < \delta' \implies \left|\frac{f(x) - f(x_0)}{x - x_0}\right| \leq M \implies |f(x) - f(x_0)| \leq M|x - x_0| < M\delta'.$$

Choose $\delta = \min\left\{\delta', \frac{\varepsilon}{M}\right\}$. Then $\mathcal{F}$ is equicontinuous on $[0,1]$. □

[24] This is deserving of its own proof, but we're hoping that you did this in class already. In some sense, it's a "follow your nose" type argument, i.e., one that you shouldn't do during a qual unless absolutely necessary.

Proof of (b). Since $[0,1]$ is a compact metric space, we will use the Arzelà–Ascoli theorem heavily: a family of functions in $C([0,1])$ is compact if and only if it is equicontinuous, closed, and pointwise bounded.

Let $f \in \mathcal{F}$. Define $f_n(x) = f(x) + n$. Then $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$. Thus $\mathcal{F}$ is not pointwise bounded. By Arzelà–Ascoli, $\mathcal{F}$ is not compact.

As $\mathcal{F}$ is equicontinuous by part (a), so is $\mathcal{F}_0$. Thus $\overline{\mathcal{F}_0}$ is closed and equicontinuous. We will show that $\mathcal{F}_0$ is pointwise bounded (and so its closure must be).

Let $f \in \mathcal{F}_0$ and $x \in [0,1]$. Define M as in the proof of part (a). By the Fundamental Theorem of Calculus,

$$f(x) = f(0) + \int_0^x f'(t)\ dt.$$

In absolute value, using the triangle inequality and our favorite integration facts,

$$|f(x)| \leq |f(0)| + \left| \int_0^x f'(t)\ dt \right| \leq 3 + \int_0^x |f'(t)|\ \ dt \leq 3 + \int_0^x M\ dt = 3 + Mx \leq 3 + M.$$

Thus $|f(x)| \leq 3 + M$ and so f is pointwise bounded. Hence $\mathcal{F}_0$ is pointwise bounded.

By Arzelà–Ascoli, $\overline{\mathcal{F}_0}$ is compact. $\square$

6

Topology

Topology Exam

Summer 2012

Problem 6.1.1

Let $p : X' \to X$ be a covering map and assume that X' is path-connected. Let $x_0, x_1 \in X'$ and $x \in X$ such that $p(x_0) = x = p(x_1)$. Prove that the subgroups $p_*\pi_1(X', x_0) = H_0$ and $p_*\pi_1(X', x_1) = H_1$ are conjugate in $\pi_1(X, x)$.

Notes and Comments

Proof. Since X' is path connected, there is a path γ' from x_0 to x_1 in X'. Projecting γ' by p gives a loop γ with base point x in X. So γ represents $g = [\gamma] \in \pi_1(X, x)$.

Note that, for any loop ℓ with base point x_0, the path $\overline{\gamma}'\ell\gamma'$ is a loop at x_1. That is $g^{-1}H_0 g \subset H_1$. By symmetry of the argument, we also have that $gH_1g^{-1} \subset H_0$. Conjugating this last relation by g^{-1} gives our desired equality. □

Problem 6.1.2

Of the following smooth manifolds, determine which ones admit a continuous nowhere vanishing vector field:

(1) S^2 **minus a point**

(2) S^2

(3) S^3

(4) $S^1 \times S^1$

(5) $SL(n, \mathbb{R})$

(6) An oriented, compact surface of genus three with no boundary.

Notes and Comments

Proof. (1) Yes. S^2 minus a point is homeomorphic to $\mathbb{R}^2$ by stereographic projection and $\mathbb{R}^2$ has many such vector fields.

(2) No by the Hairy Ball Theorem.

(3) Yes by the Hairy Ball Theorem.

(4) Yes. Apply the Hairy Ball Theorem to each factor and take the product vector field.

(5) Yes. $SL(n, \mathbb{R})$ is a Lie group and thus admits a global frame.

(6) No. This is a consequence of the Poincaré–Hopf Theorem. If such a vector field exists, then the the Euler characteristic of the surface must be zero. However, the Euler characteristic of this surface is -4.

□

Problem 6.1.3

Denote by S^n the unit sphere in $\mathbb{R}^{n+1}$. If $F : S^n \to S^n$ is the antipodal map defined by $F(x) = -x$, then show by calculation that the degree of F is $(-1)^{n+1}$.

Notes and Comments

Proof. Note that, viewed as a map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, the map F simply negates each coordinate of x. Thus F can be written as the composition of $n+1$ reflections across the hyperplanes perpendicular to the standard basis vectors. The degree of a composition of maps is equal to the product of the degrees, hence it suffices to recall that the degree of a reflection is -1. □

Problem 6.1.4

Determine the singular homology groups of the standard torus (i.e., regarded as an identification space of a 2-dimensional rectangle) using the Mayer–Vietoris sequence.

Notes and Comments

Proof. To apply the Mayer–Vietoris construction we need to decompose $\mathbb{T}^2$ into two open overlapping sets. We can decompose $\mathbb{T}^2$ into two overlapping cylinders A and B by imagining our torus in its standard doughnut orientation and removing the left third (for A) or right third (for B). Then A and B are both homeomorhpic to S^1 and $A \cap B$ is homeomorphic to $S_1 \sqcup S^1$.

Using the homology of spheres we have that

$$H_n(A) = H_n(B) = \begin{cases} \mathbb{Z} & n \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \text{ and } H_n(A \cap B) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

In order to compute the homology groups of $\mathbb{T}^2$, we consider individual segments of the Mayer–Vietoris long exact sequence. When $n > 2$ we have

$$H_n(A) \oplus H_n(B) \longrightarrow H_n(\mathbb{T}^2) \longrightarrow H_{n-1}(A \cap B)$$

We know that the outside groups are trivial and thus, by exactness, $\boxed{H_n(\mathbb{T}^2) = 0 \text{ for } n > 2}$. To compute $H_2(\mathbb{T}^2)$, we begin with the next portion of the sequence:

$$\begin{tikzcd}
H_2(A) \oplus H_2(B) \arrow[r] \arrow[d, equal] & H_2(\mathbb{T}^2) \arrow[r, "\partial"] & H_1(A \cap B) \arrow[r, "i_*"] \arrow[d, equal] & H_1(A) \oplus H_1(B) \arrow[d, equal] \\
0 & & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z}
\end{tikzcd}$$

By exactness of the sequence, we know that ∂ is injective and so our desired group $H_2(\mathbb{T}^2) \cong \operatorname{im}(\partial)$. As the sequence is exact, this is isomorphic to the kernel of i_* (which is induced by the inclusions of $A \cap B$ into A and B). To construct this map explicitly, we consider the two generators of $H_1(A \cap B)$ and notice that, after inclusion into A, B, they must map to the same element. Thus i_* takes the form:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and has kernel equal to $\mathbb{Z}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ which is isomorhpic to $\mathbb{Z}$. Thus $\boxed{H_2(\mathbb{T}^2) \cong \mathbb{Z}}$.

We take a similar approach to computing $H_1(\mathbb{T}^2)$. Beginning with the groups that we know in the sequence gives:

$$H_1(A \cap B) \xrightarrow{i_*} H_1(A) \oplus H_1(B) \xrightarrow{p} H_1(\mathbb{T}^2) \xrightarrow{\partial} H_0(A \cap B) \longrightarrow H_0(A) \oplus H_0(B)$$

Taking generators for $H_0(A \cap B)$ again gives that the map i_* is represented by $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Thus i_* has kernel and image both isomorphic to $\mathbb{Z}$. Extracting the relevant short exact sequence, we obtain:

$$0 \longrightarrow \ker(\partial) \longrightarrow H_1(\mathbb{T}^2) \longrightarrow \operatorname{im}(\partial) \longrightarrow 0$$

As above, we can compute $\operatorname{im}(\partial) \cong \ker(i_*) \cong \mathbb{Z}$. To characterize $\ker(\partial)$ we use exactness to note that $\ker(\partial) \cong \operatorname{im}(p) \cong \mathbb{Z}^2/\ker(p) \cong \mathbb{Z}^2/\operatorname{im}(i_*) \cong \mathbb{Z}$, where the last steps follow from our explicit computations of i_*. Thus, our short exact sequence becomes

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_1(\mathbb{T}^2) \longrightarrow \mathbb{Z} \longrightarrow 0$$

which splits since $\mathbb{Z}$ is free. Hence $\boxed{H_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}}$.

Finally, since $\mathbb{T}^2$ is connected, we know that $H_0(\mathbb{T}^2) = \mathbb{Z}$. Thus, our final answer is

$$H_n(\mathbb{T}^2) = \begin{cases} \mathbb{Z} & n \in \{0, 2\} \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

$\square$

Problem 6.1.5

Let φ_1 and φ_2 be charts on $\mathbb{R}$ defined by $\varphi_1(t) = t$ and $\varphi_2(t) = t^3$. Are they C^∞ compatible? Prove your answer.

Notes and Comments

Proof. In order to be C^∞ compatible we must have that $\varphi_1(\varphi_2^{-1}(t))$ is smooth on $\mathbb{R}$. Unfortunately, $\varphi_1(\varphi_2^{-1}(t)) = \sqrt[3]{t}$ is not differentiable at 0. Thus they are not C^∞ compatible. □

Problem 6.1.6

Define the wedge product of two differential forms on a manifold. How does one use this operation to define the cup product of two de Rham cohomology classes? Prove that the cup product is well defined.

Notes and Comments

Proof. The wedge product is the product in the exterior algebra defined as the quotient of the tensor algebra by the square ideal $I = \langle\{x \otimes x : x \in \mathcal{A}\}$. Thus the wedge of two differential forms is

$$\boxed{u \wedge v := u \otimes v \pmod{I}}.$$

To define the cup product on de Rham cohomology classes, we note that the classes are formed from an equivalence relation on differential forms (cocycles mod coboundaries) and define $\boxed{[u] \smile [v] := [u \wedge v]}$. To show that this operation is well-defined, let $[u] \in \Omega^n_{dR}, [v] \in \Omega^m_{dR}$ be cohomology classes with $u, u' \in [u]$ and $v, v' \in [v]$. Thus there are coboundaries $\partial x, \partial y$ such that

$$u = u' + \partial x,\ v = v' + \partial y.$$

We show that $[u \wedge v] = [u' \wedge v']$. Computing using the properties of $\wedge$, we have

$$\begin{aligned}
u \wedge v =& (u' + \partial x) \wedge (v' + \partial y) \\
=& u' \wedge v' + u' \wedge \partial y + \partial x \wedge v' + \partial x \wedge \partial y \\
=& u' \wedge v' + (u' \wedge \partial y + (-1)^n \underbrace{\partial u'}_{=0} \wedge y) + (\partial x \wedge v' + (-1)^m x \wedge \underbrace{\partial v'}_{=0}) + (\partial x \wedge \partial y + (-1)^m \underbrace{\partial\partial x}_{=0} \wedge \partial y) \\
=& u' \wedge v' + \partial((-1)^n u' \wedge y + x \wedge v' + (-1)^m \partial x \wedge y)
\end{aligned}$$

Taking cohomology classes at both ends of this chain of equalities gives the desired result. □

Topology Exam

Fall 2012

Problem 6.2.1

Let M be a smooth manifold, and let $x^1, \ldots, x^n$ be a local coordinate system defined on an open set $U \subseteq M$. Consider the $(1,1)$-tensor field C defined on U in local coordinates by

$$C = \sum_{i=1}^{n} dx^i \otimes \frac{\partial}{\partial x^i}.$$

Show that C is independent of the choice of local coordinates and hence defines a smooth global tensor field on M.

Notes and Comments

Proof. Let $(y^1, ..., y^n)$ be another local coordinate system for M about a point $p \in U$. Pushing forward the change of coordinate formula in $\mathbb{R}^n$ to a change of basis formula for T_pM, we get the following formula:

$$\left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^{n} \left.\frac{\partial y^j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^j}\right|_p.$$

Since dx^i_p and dy^j_p are dual to $\left.\frac{\partial}{\partial x^i}\right|_p$ and $\left.\frac{\partial}{\partial y^j}\right|_p$, respectively, we get the additional formula:

$$\begin{aligned} dy^k_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= dy^k_p\left(\sum_{j=1}^{n} \left.\frac{\partial y^j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^j}\right|_p\right) \\ &= \sum_{j=1}^{n} \left.\frac{\partial y^j}{\partial x^i}\right|_p dy^k_p\left(\left.\frac{\partial}{\partial y^j}\right|_p\right) \\ &= \sum_{j=1}^{n} \left.\frac{\partial y^j}{\partial x^i}\right|_p \delta^k_j \\ &= \left.\frac{\partial y^k}{\partial x_i}\right|_p. \end{aligned}$$

Hence

$$dy^k_p = \sum_{i=1}^{n} \left.\frac{\partial y^k}{\partial x_i}\right|_p dx^i_p.$$

Since the change of coordinate function is a diffeomorphism between neighborhoods of $(x^i(p))$ and $(y^i(p))$ in $\mathbb{R}^n$, the matrix $\left[\left.\frac{\partial y^k}{\partial x_i}\right|_p\right]_{(k,i)}$ is invertible. Let this matrix be denoted by $J(p)$ with inverse $J^{-1}(p)$.

Then we can carry out the following substitutions:

$$\begin{aligned} C|_p &= \sum_{i=1}^{n} dx_p^i \otimes \frac{\partial}{\partial x^i}\Big|_p \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} J_{i,k}^{-1}(p) dy_p^k \right) \otimes \left(\sum_{j=1}^{n} J_{j,i}(p) \frac{\partial}{\partial y^j}\Big|_p \right) \\ &= \sum_{j,k=1}^{n} \left(\sum_{i=1}^{n} J_{j,i}(p) J_{i,k}^{-1}(p) \right) dy_p^k \otimes \frac{\partial}{\partial y^j}\Big|_p \\ &= \sum_{j,k=1}^{n} \delta_{j,k} dy_p^k \otimes \frac{\partial}{\partial y^j}\Big|_p \\ &= \sum_{j=1}^{n} dy_p^j \otimes \frac{\partial}{\partial y^j}\Big|_p . \end{aligned}$$

Therefore the definition of C is independent of the choice of local coordinates and thus can be used to define a global tensor field on M. □

Problem 6.2.2

Determine the critical points of the determinant mapping $\det : M_n(\mathbb{R}) \to \mathbb{R}$ defined on the space of $n \times n$ matrices. [Hint: The determinant is multilinear as a function of the columns of a matrix.]

Notes and Comments

Proof. Recall that the determinant of $X \in M_n(\mathbb{R})$ can be computed by expanding along the jth column of X via

$$\det X = \sum_{i=1}^{n} (-1)^{i+j} x^{ij} \det(X_{ij})$$

where X_{ij} is the (i, j)-minor of X. Then for $A = (a^{ij}) \in M_n(\mathbb{R})$, we have

$$\frac{\partial \det}{\partial x^{ij}}(A) = (-1)^{i+j} \det(A_{ij}).$$

Recall that critical points occur whenever the derivative (pushforward) of a map is 0 (in our case, $\det_{*,A} = 0$). Since we're working in Euclidean space ($M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$), the pushforward is computed via the matrix of partial derivatives (i.e., the Jacobian matrix). Hence, by our work above

$$\begin{aligned} \det{}_{*,A} = 0 &\Leftrightarrow \det(A_{ij}) = 0 \ \forall i, j \\ &\Leftrightarrow \operatorname{rank} A < n - 1. \end{aligned}$$

To see this, note that $\operatorname{rank} A < n-1$ means that each (i, j)-minor is also not invertible. That is, the critical points of det are precisely $\{A \in M_n(\mathbb{R}) \mid \operatorname{rank} A < n - 1\}$. □

Problem 6.2.3

Let $S \subseteq \mathbb{R}^3$ be the surface with boundary given by

$$S = \left\{(x, y, z) \ : \ z = x^2 + y^2, z \leq 9\right\},$$

oriented by the unit normal field $N = (n_1, n_2, n_3)$ with $n_3 < 0$. Let ω be the 2-form on $\mathbb{R}^3$ given by

$$\omega = e^{z \sin y} dy \wedge dz + \tan^{-1}(x \sinh z) dz \wedge dz + 2dx \wedge dy.$$

Compute the integral $\int_S \omega$.

Notes and Comments

Proof. This problem requires one very important observation: since the cone S is diffeomorphic (just need smoothly homotopic) to the disk $D = \{(x, y, 9) \mid x^2 + y^2 \leq 9\}$, S must have trivial de Rham cohomology in degree 2. Thus every closed 2-form is exact.

Note that $d\omega = 0$ since each term acquires a repeat dx, dy, or dz and thus is 0 (or, as in the last term, it simply vanishes). Hence ω is a closed 2-form which, by the remark above, means that $\omega = d\eta$ for some 1-form η.

By Stokes' theorem,

$$\int_S \omega = \int_{\partial S} \eta = \int_D \omega$$

since S and D share the same boundary. On D, $\omega = 2dx \wedge dy$ because z is constant. Hence

$$\int_S \omega = \int_D 2dx \wedge dy = 2\,\text{Area}(D) = 2(9\pi) = 18\pi.$$

Note that the orientation induced by N did not come into play *because* it's the outward unit normal vector field for S. $\square$

Problem 6.2.4

Suppose that a space X is the disjoint union $X = U \sqcup V$ of two open subspaces U and V.

(a) Use the Eilenberg–Steenrod axioms to prove that, for any homology theory, the homology groups of X are given in terms of those of U and V by

$$H_q(X) = H_q(U) \oplus H_q(V).$$

(b) Why is this result easier if we take the homology theory to be singular homology?

Notes and Comments

Proof of (a). See the solution given in problem 4 of the Summer 2014 exam (6.5.4). $\square$

Proof of (b). Recall that the continuous image of a connected set must be connected. Hence the images of the basis elements (for singular homology) land in either X or Y and so the homology must split. $\square$

Problem 6.2.5

Let $p : Y \to X$ be a covering map. Let Z be any connected space, and let $f : Z \to X$ be a continuous map. Suppose that $f_1 : Z \to Y$ and $f_2 : Z \to Y$ are continuous lifts of f (i.e., $p \circ f_i = f$ for $i = 1, 2$) that agree at some point $z_0 \in Z$. Show that $f_1 = f_2$ on all of Z.

Notes and Comments

Proof. Let $A = \{z \in Z \mid f_1(z) = f_2(z)\}$. We will show that A is clopen.

Let $z \in Z$ and let U be an evenly covered neighborhood of $f(z)$. Then $p^{-1}(U) = \displaystyle\bigsqcup_{\alpha \in I} U_\alpha$ where $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism.[1] Then $\boxed{f_1(z) \in U_{\alpha_1}}$ and $\boxed{f_2(z) \in U_{\alpha_2}}$ for some $\alpha_1, \alpha_2 \in I$.

Let $V_i = f_i^{-1}(U_{\alpha_i})$. Since f_i is continuous, V_i is open in Z. Define $V = V_1 \cap V_2$ which is necessarily a neighborhood of z. Moreover, $\boxed{f_i(V) \subset U_{\alpha_i}}$ for $i = 1, 2$.

A **is open:** If $z \in A$ then $f_1(z) = f_2(z)$ and so $U_{\alpha_1} = U_{\alpha_2}$. For any $y \in V$, we have

$$p \circ f_1(y) = f(y) = p \circ f_2(y).$$

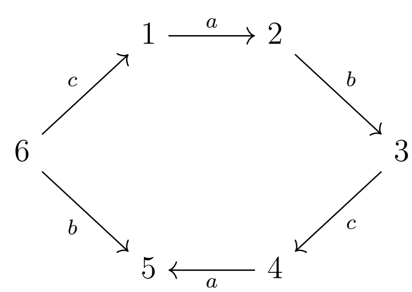
Since p is injective on U_{α_1}, this means $f_1(y) = f_2(y)$ and thus $y \in A$. Hence $f_1|_V = f_2|_V$. That is, V is an open set such that $z \in V \subset A$. So $\boxed{A \text{ is open in } Z}$.

A **is closed:** If $z \in Z \setminus A$ then $f_1(z) \neq f_2(z)$ and so $\alpha_1 \neq \alpha_2$. Thus $U_{\alpha_1} \cap U_{\alpha_2} = \varnothing$. That is, $f_1(y) \neq f_2(y)$ for all $y \in V$. Hence V is an open set such that $z \in V \subset Z \setminus A$. So $Z \setminus A$ is open in Z. That is, $\boxed{A \text{ is closed}}$.

As A is clopen in Z and Z is connected, $A = \varnothing$ or $A = Z$. We assumed that $z_0 \in A$ and so $A = Z$. That is, $f_1 = f_2$ on Z as desired. $\square$

Problem 6.2.6

Consider the space X obtained as the quotient space of a planar hexagon and its interior by identifying boundary edges of the hexagon in pairs according to the following scheme:



Compute the homology groups of X.

Notes and Comments

Proof. This cute problem has a very short solution if you can identify X. Note that, by relabeling based

[1] Here's our stack of pancakes! Eat up!

on the end points of our edges, we obtain

$$\begin{tikzcd}
& 1 \arrow[r, "a"] & 2 \arrow[dr, "b"] & \\
2 \arrow[ur, "c"] \arrow[dr, "b"'] & & & 2 \arrow[dl, "c"] \\
& 2 & 1 \arrow[l, "a"] &
\end{tikzcd}.$$

Possibility 1: Note that the edges ca repeat in the same order (having the same orientation). This implies that they can be collapsed to a single new edge d and so we have

$$\begin{tikzcd}
2 \arrow[r, "d"] \arrow[d, "b"'] & 2 \arrow[d, "b"] \\
2 & 2 \arrow[l, "d"]
\end{tikzcd}$$

This space is better known as the Klein bottle and has homology groups

$$H_q(X) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & q = 1 \end{cases}.$$

Possibility 2: Under stress and feeling as though we should write in more details, we might miss out on the first possibility. However, not all is lost: this CW-complex for X has 2 vertices, 3 edges, and a face. Let's use cellular homology!

The cellular chain groups W_q of X are

$$\ldots \longrightarrow 0 \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0$$

where $W_0 = \mathbb{Z}[1,2]$, $W_1 = \mathbb{Z}[a,b,c]$, and $W_2 = \mathbb{Z}[f]$ where f is oriented counterclockwise. With respect to these bases, the boundary maps have matrix representations

$$[\partial_0] = \begin{pmatrix} 0 & 0 \end{pmatrix}, [\partial_1] = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}, [\partial_2] = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}.$$

The Smith normal form of these maps are, respectively,

$$\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

Hence we have

$$\begin{aligned}
H_0(X) &= \frac{\ker \partial_0}{\operatorname{im} \partial_1} \cong \frac{\mathbb{Z}^2}{\mathbb{Z}} \cong \mathbb{Z} \\
H_1(X) &= \frac{\ker \partial_1}{\operatorname{im} \partial_2} \cong \frac{\mathbb{Z}^2}{2\mathbb{Z}} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\
H_2(X) &= \frac{\ker \partial_2}{\operatorname{im} \partial_3} \cong 0
\end{aligned}$$

That is, $H_q(X)$ is as above for all q. $\square$

Topology Exam

Summer 2013

Problem 6.3.1

Prove that the Lie bracket of two vector fields is a vector field.

Notes and Comments

Proof. Let V and W be smooth vector fields on M. It's enough to show that $[V,W]$ is a derivation. Let $f, g \in C^\infty(M)$. Then, since V and W are derivations,

$$\begin{aligned} V(W(fg)) &= V(fW(g) + W(f)g) \\ &= V(fW(g)) + V(W(f)g) \\ &= fV(W(g)) + V(f)W(g) + W(f)V(g) + V(W(f))g. \end{aligned}$$

Similarly,

$$W(V(fg)) = fW(V(g)) + W(f)V(g) + V(f)W(g) + W(V(f))g.$$

Hence

$$\begin{aligned} [V,W](fg) = V(W(fg)) - W(V(fg)) &= fV(W(g)) + V(W(f))g - fW(V(g)) - W(V(f))g \\ &= f(VW(g) - WV(g)) + (VW(f) - WV(f))g \\ &= f[V,W](g) + [V,W](f)g. \end{aligned}$$

Thus $[V,W]$ is a derivation and hence a smooth vector field. □

Problem 6.3.2

If $1 \le n < m$, show that no open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$.

Notes and Comments

Proof. Let $U^{\text{open}} \subseteq \mathbb{R}^n$ and $V^{\text{open}} \subseteq \mathbb{R}^m$. Assume that φ is a homeomorphism. Then, at $p \in U$ and $\varphi(p)$, we may compute the local homologies of U and V. Since local homology is local,

$$H_q(U \mid p) \cong H_q(\mathbb{R}^n \mid p) = \begin{cases} \mathbb{Z} & q = n \\ 0 & \text{else} \end{cases}.$$

Thus, since φ is a homeomorphism, $H_q(V \mid \varphi(p)) \cong H_q(\mathbb{R}^m \mid \varphi(p))$ must agree with that of U at p. That is, $n = q = m$. □

Problem 6.3.3

(a) Does there exist a manifold whose boundary is the disjoint union of two Klein bottles? Construct such a manifold or prove it does not exist.

(b) Does there exist an orientable manifold whose boundary is the disjoint union of two Klein bottles? Construct such a manifold or prove it does not exist.

(c) Does there exist a Lie group whose boundary is a torus $S^1 \times S^1$? Construct such a Lie group or prove it does not exist.

Notes and Comments

Proof. For (a), let K be the Klein bottle. Then $K \times [0,1]$ is a manifold with boundary $\partial(K \times [0,1]) = K \times \partial[0,1] = K \times \{0\} \cup K \times \{1\}$. Thus we have such a manifold.

However, (b) is false. If M is orientable then ∂M must be as well. Since the Klein bottle is non-orientable, it (and thus unions of copies of it) cannot be the boundary of an orientable manifold.

Additionally, (c) is incredibly false. Lie groups have no boundary and thus the torus[2] cannot be the boundary of a Lie group. □

Problem 6.3.4

Let G be a topological group; that is, G is a group equipped with a topology such that multiplication $\mu : G \times G \to G$ and inversion $\iota : G \to G$ are continuous. Show that the fundamental group $\pi_1(G, e)$ is abelian.

Notes and Comments

Proof. Let σ and τ be loops at e. Define, for $s \in [0,1]$,

$$f_s(t) = \begin{cases} \sigma\left(\frac{2t}{1+s}\right) & \text{if } 0 \le t \le \frac{s+1}{2} \\ e & \text{if } \frac{s+1}{2} \le t \le 1 \end{cases} \text{ and } g_s(t) = \begin{cases} e & \text{if } 0 \le t \le \frac{1-s}{2} \\ \tau\left(\frac{2t+s-1}{1+s}\right) & \text{if } \frac{1-s}{2} \le t \le 1 \end{cases}.$$

Intuitively, f_s is a homotopy from $\sigma * e$ to σ and g_s is a homotopy from $e * \tau$ to τ. While the definitions of f_s and g_s are seemingly opaque, we construct f_s in the Notes for this problem as an example.

Since multiplication is continuous, $f_s g_s$ is a homotopy between $\sigma * \tau$ and $\sigma\tau$. Thus $[\sigma][\tau] = [\sigma\tau]$ (for any pair of loops). Moreover, $g_s f_s$ is a homotopy between $\sigma * \tau$ and $\tau\sigma$. Hence $[\sigma][\tau] = [\tau\sigma] = [\tau][\sigma]$, and so $\pi_1(G, e)$ is abelian. □

Problem 6.3.5

Prove that the wedge product of differential forms gives a well-defined operation on the cohomology groups of the manifold. (This operation is called the cup product of cohomology classes.)

Notes and Comments

Proof. See the solution to problem 6 of the Summer 2012 exam (6.1.6). □

Problem 6.3.6

Suppose that A and B are subspaces of X and that B is a deformation retract of A. Show that $H_q(X, B) \cong H_q(X, A)$ for all $g \ge 0$. (You may use the 5-lemma without proof.)

Notes and Comments

[2] Being a surface of sound mind.

Proof. Since B is a deformation retract of A, the homology groups are isomorphic via the inclusion $i : B \to A$. Consider the long exact sequences associated to (X, A) and (X, B):

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(B) \arrow[r] \arrow[d, "i_*"] & H_q(X) \arrow[r] \arrow[d, "\mathrm{Id}_*"] & H_q(X,B) \arrow[r] \arrow[d, "f"] & H_{q-1}(B) \arrow[r] \arrow[d, "i_*"] & H_{q-1}(X) \arrow[r] \arrow[d, "\mathrm{Id}_*"] & \cdots \\
\cdots \arrow[r] & H_q(A) \arrow[r] & H_q(X) \arrow[r] & H_q(X,A) \arrow[r] & H_{q-1}(A) \arrow[r] & H_{q-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

Here, the map $f : (X, B) \to (X, A)$ denotes the inclusion of pairs. Since all the non-boundary maps involved in this diagram are inclusions, we get lovely commuting squares. Since i_* and Id_* are isomorphisms (on both sides), the Five Lemma states that f must also be an isomorphism. $\square$

Topology Exam

Fall 2013

Problem 6.4.1

Let X and Y be topological spaces with $x_0 \in X$ and $y_0 \in Y$. Let $X \times Y$ have the product topology. Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Notes and Comments

Proof. Since $X \times Y$ has the product topology, the projection maps

$$\pi_X : X \times Y \to X \text{ and } \pi_Y : X \times Y \to Y$$

are continuous and thus induce group homomorphisms

$$\pi_{X*} : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \text{ and } \pi_{Y*} : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0).$$

By the universal property of the product, there exists a unique map f such that the diagram

$$\begin{tikzcd}
 & \pi_1(X \times Y, (x_0, y_0)) \arrow[dl, "\pi_{X*}"'] \arrow[d, dashed, "f"] \arrow[dr, "\pi_{Y*}"] & \\
\pi_1(X, x_0) & \pi_1(X, x_0) \times \pi_1(Y, y_0) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & \pi_1(Y, y_0)
\end{tikzcd}$$

commutes. Observe that for a path $u \in \pi_1(X \times Y, (x_0, y_0))$,

$$f([u]) = ([\pi_X(u)], [\pi_Y(u)]).$$

It will suffice to show that f is injective and surjective.

Injective: Let $[u] \in \mathrm{Ker}(f)$ be represented by $u(t) = (p_x(t), p_y(t))$. Then $[\pi_X(u)] = [c_{x_0}]$ and $[\pi_Y(u)] = [c_{y_0}]$.

Note that $\pi_X(u)(t) = p_x(t)$ and there is a homotopy $h_X : I \times I \to X$ between p_x and c_{x_0}. Similarly $\pi_Y(u)(t) = p_y(t)$ and there is a homotopy h_Y between p_y and c_{y_0}.

Define a continuous function $h : I \times I \to X \times Y$ by $h(s,t) = (h_X(s,t), h_Y(s,t))$ is continuous. Then we have

$$h(0,t) = (p_x(t), p_y(t)) = u(t), \;\; h(1,t) = (x_0, y_0), \text{ and } h(s,0) = h(s,1) = (x_0, y_0).$$

Hence u is homotopic to $c_{(x_0,y_0)}$ via h. Thus $[u] = [c_{(x_0,y_0)}]$ and so f is injective.

Surjective: Let $[u_X] \in \pi_1(X, x_0)$ and $[u_Y] \in \pi_1(Y, y_0)$. Then $(u_X, u_Y) : [0,1] \to X \times Y$ is a loop in $X \times Y$ based at (x_0, y_0). Moreover,

$$f([(u_X, u_Y)]) = (\pi_{X*}([(u_X, u_Y)]), \pi_{Y*}([(u_X, u_Y)])) = ([\pi_X \circ (u_X, u_Y)], [\pi_Y \circ (u_X, u_Y)]) = ([u_X], [u_Y]).$$

Thus f is surjective.

Hence f is an isomorphism and thus $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ as desired. □

Problem 6.4.2

Let M be a smooth manifold, X a continuous vector field on M (i.e., a continuous section of the tangent bundle TM). There are two reasonable definitions of what it means for X to be smooth at a point p in M:

(a) Definition 1: Let (x, U) be a local coordinate system defined on an open neighborhood U of p; then X can be expressed in local coordinates as $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ for some real-valued functions $a^1, \ldots, a^n$ defined on U. Then X is *smooth* at p provided that each coordinate function a^i is smooth at p.

(b) Definition 2: The vector field X is *smooth* at p if, for every smooth function f defined on a neighborhood of p, the function $X(f)$ is smooth at p.

Notes and Comments

Proof. $\underline{(a) \Rightarrow (b)}$: Assume f is a smooth function defined on a neighborhood V of p. Let (x, U) be a chart near p with $U \subseteq V$.[3] Then, in coordinates,

$$X(f) = \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^i}.$$

Since a^i is smooth at p by (a) and f is smooth on $V \ni p$, $X(f)$ is the sum of products of functions smooth at p. That is, $X(f)$ is smooth at p.

$\underline{(b) \Rightarrow (a)}$: Let (x, U) be a chart near p. Then the coordinate maps x^j are smooth functions on U for all $j = 1, \ldots, n$.[4] so $X(x^j)$ is smooth at p by (b). That is, letting δ_{ij} be the Kronecker delta function,

$$X(x^j) = \sum_{i=1}^{n} a^i \frac{\partial x^j}{\partial x^i} = \sum_{i=1}^{n} a^i \delta_{ij} = a^j.$$

That is, a^j is smooth at p. So each of the coordinate maps is smooth at p as desired. □

Problem 6.4.3

Show that S^{n-1} is not a retract of $E^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ for $n \geq 1$. Use this to prove the Brouwer Fixed-Point Theorem; that is, show that if $n \geq 1$, then any continuous map $f : E^n \to E^n$ must have a fixed point.

Notes and Comments

Proof. Assume $r : E^n \to S^{n-1}$ is a retraction of $i : S^{n-1} \hookrightarrow E^n$. Then $r \circ i = \mathrm{id}_{S^{n-1}}$. On the level of homology[5], we have

$$\mathrm{id}_{S^{n-1}*} = (r \circ i)_* = r_* \circ i_*.$$

[3] We may assume $U \subseteq V$ since, otherwise, we could simply intersect U with V.

[4] Technically this function isn't defined on U but rather on $x(U)$. However, in the great tradition of differential topology, we conflate the functions x^j and $x^j \circ x^{-1}$ because it's "easier."

[5] We use reduced homology specifically because we don't want to single out the case $n = 1$. Laziness wins.

Thus we have a commuting diagram (considered in degree $n-1 \geq 0$):

$$\begin{tikzcd} \mathbb{Z} = \widetilde{H}_{n-1}(S^{n-1}) \arrow[r, "i_*"] \arrow[dr, "\mathrm{id}_{S^{n-1}*}"'] & \widetilde{H}_{n-1}(E^n) = 0 \arrow[d, "r_*"] \\ & \widetilde{H}_{n-1}(S^{n-1}) = \mathbb{Z} \end{tikzcd}$$

That is, $\mathrm{Id}_{S^{n-1}*}$ factors through the 0 map (r_*), but this is impossible. $\lightning$ Hence E^n has no retraction onto S^{n-1}. $\square$

Proof of the Fixed-Point Theorem. Assume $f : E^n \to E^n$ has no fixed point. For $x \in E^n$, define $r(x)$ to be the intersection of the ray from $f(x)$ to x with S^{n-1}. That is, $r(x)$ is the point of $(1-t)f(x) + tx$ such that $\|(1-t)f(x) + tx\| = 1$. In particular, $r|_{S^{n-1}} = \mathrm{Id}_{S^{n-1}}$. We know that $r : E^n \to S^{n-1}$ is continuous and thus r is a retraction. However, no such retraction may exist. $\lightning$ Hence f has a fixed point. $\square$

Problem 6.4.4

(a) Does a boundary of a parallelizable manifold have to be a parallelizable manifold? Prove your answer.

(b) Does a product of parallelizable manifolds have to be a parallelizable manifold? Prove your answer.

(c) Is the Klein bottle a parallelizable manifold? How about the torus $S^1 \times S^1$? Prove your answer.

Notes and Comments

Proof. (a) $\boxed{\text{False}}$ Consider the compact orientable 3-manifold better known as the 3-ball, B^3. This manifold is parallelizable but its boundary is S^2, which admits no global vector field (let alone frame) by the Hairy Ball Theorem.

(b) $\boxed{\text{True}}$ Let M and N be manifolds with global frames $(E_i)_{i=1}^{\dim M}$ and $(F_j)_{j=1}^{\dim N}$ respectively. Then $\{E_1, \ldots, E_{\dim M}, F_1, \ldots, F_{\dim N}\}$ is a global frame for $M \times N$.

(c) $\boxed{\text{False, True}}$ Since any parallelizable manifold must be orientable, the Klein bottle's lack of orientability prohibits being parallelizable. However, the torus is parallelizable since it is a Lie group (alternatively, each S^1 factor has a global framing $\dfrac{\partial}{\partial \theta_i}$ and we may apply (b)).

$\square$

Problem 6.4.5

Let $n \geq 2$ and $B \subset S^n$ a wedge of two spheres; that is, B is a closed subspace of S^n homeomorphic to a figure eight so that $B = C \cup D$ with C and D homeomorphic to S^1 and $C \cap D$ a single point. Compute $H_q(S^n \setminus B)$ for $q \geq 0$.

Notes and Comments

Theorem 6.4.1 *With all notation as above we have for $n \geq 2$*

$$H_q(S^n \setminus B) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } q = n-2 > 0 \\ \mathbb{Z} & \text{if } q = 0 \text{ and } n > 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if } q = 0 \text{ and } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

To prove this we will use two tools, the Mayer–Vietoris exact sequence and the following characterization of reduced homology of $S^n \setminus S^k$.

Theorem 6.4.2 (Theorem 6.3, Massey pg. 214): *Let A be a subset of S^n which is homeomorphic to S^k, $0 \leq k \leq n-1$. Then $\widetilde{H}_{n-k-1}(S^n \setminus A) = \mathbb{Z}$ and $\widetilde{H}_q(S^n \setminus A) = 0$ if $q \neq n-k-1$.*

Proof of Theorem 6.4.1. We will separate this into three cases, (1) $q > 0$, (2) $q = 0$ and $n > 2$, and (3) $q = 0$ and $n = 2$. In all cases we will use the fact that we can write $S^n \setminus B$ as an intersection of spaces that Theorem 6.4.2 already tells us about. To this end let $W_1 = S^n \setminus C$ and $W_2 = S^n \setminus D$, so W_1, W_2 are n-spheres with a circle deleted. Also $W_1 \cap W_2 = S^n \setminus B$ is the space we're actually aiming for. Finally $W_1 \cup W_2$ is just S^n with a single point removed so it has very simple homology (it's a contractible space).

Mayer–Vietoris gives the long exact sequence

$$\cdots \to H_{q+1}(W_1 \cup W_2) \to H_q(S^n \setminus B) \to H_q(W_1) \oplus H_q(W_2) \to H_q(W_1 \cup W_2) \to \cdots$$

and we will focus on certain finite exact sequences that can be obtained from this long exact sequence.

Case 1: ($q > 0$) For this case we focus on the finite exact sequence

$$H_{q+1}(W_1 \cup W_2) \to H_q(S^n \setminus B) \to H_q(W_1) \oplus H_q(W_2) \to H_q(W_1 \cup W_2).$$

Since $q > 0$ and $W_1 \cup W_2$ is contractible we know $H_{q+1}(W_1 \cup W_2) \cong H_q(W_1 \cup W_2) \cong 0$, meaning that the middle map above is an isomorphism and so

$$H_q(S^n \setminus B) \cong H_q(W_1) \oplus H_1(W_2).$$

But $H_q(W_1) \cong H_q(W_2) \cong \mathbb{Z}$ if $q = n-1-1 = n-2$ by Theorem 6.4.2. If however $q \neq n-2$ then $H_q(W_1) \cong H_q(W_2) \cong 0$. This verifies the result for $q > 0$.

Case 2: ($n > 2, q = 0$) For this case we first note that the Mayer–Vietoris sequence is slightly longer (there is 0 at the end is the only difference).

$$H_1(W_1 \cup W_2) \to H_0(S^n \setminus B) \to H_0(S^n \setminus W_1) \oplus H_0(S^n \setminus W_2) \to H_0(W_1 \cup W_2) \to 0,$$

and again we use the fact that $W_1 \cup W_2$ is contractible to note that $H_1(W_1 \cup W_2) \cong 0$ and $H_0(W_1 \cup W_2) \cong \mathbb{Z}$. Since $n > 2$ we must have $0 \neq n-2$, meaning that $H_0(S^n \setminus W_1) \cong H_0(S^n \setminus W_2) \cong \mathbb{Z}$ by Theorem 6.4.2. Then rewriting this exact sequence we get

$$0 \to H_0(S^n \setminus B) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

All three of these groups are abelian, so the rank theorem for short exact sequences of abelian groups means that the rank of $H_0(S^n \setminus B)$ must be $2-1=1$. But of course $H_0(S^n \setminus B)$ is actually a free abelian group, so $H_0(S^n \setminus B) = \mathbb{Z}$.

Case 3: ($n = 2, q = 0$) The only difference now is that $H_0(S^2 \setminus W_1) = H_0(S^2 \setminus W_2) = \mathbb{Z} \oplus \mathbb{Z}$ (again by Theorem 6.4.2) and the short exact sequence we obtain for the homology group in question is now

$$0 \to H_0(S^n \setminus B) \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

Again though the rank theorem tells us that the rank of $H_0(S^2 \setminus B)$ must be $4-1=3$, and since this group is free abelian we have $H_0(S^2 \setminus B) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. $\square$

Problem 6.4.6

(a) Let $\varphi : S^2 \to \mathbb{R}^{17}$ be a smooth map. Let ω be a closed 2-form on $\mathbb{R}^{17}$. Compute the integral $\int_{S^2} \varphi^*\omega$.

(b) Let $\varphi : S^3 \to S^2$ and $\psi : S^2 \to S^4$ be smooth maps of oriented manifolds. Let ω be a 3-form on S^4. Compute $\int_{S^3} (\psi \circ \varphi)^*\omega$.

Notes and Comments

Proof of (a). Since ω is a closed 2-form on $\mathbb{R}^{17}$ and $H_dR^2(\mathbb{R}^{17}) = 0$, ω is an exact 2-form. Thus $\omega = d\eta$. Hence, since d commutes with "everything" (†) and Stokes' Theorem ($*$),

$$\int_{S^2} \varphi^*\omega = \int_{S^2} \varphi^*(d\eta) \overset{(\dagger)}{=} \int_{S^2} d\varphi^*\eta \overset{(*)}{=} \int_{\partial S^2} \varphi^*\eta = \int_{\varnothing} \varphi^*\eta = 0.$$

That is, $\boxed{\int_{S^2} \varphi^*\omega = 0}$. □

Proof of (b). Since S^2 is 2-dimensional, $\Omega^3(S^2) = 0$. Since $\varphi^*\omega$ is a 3-form on S^2, $\varphi^*\omega = 0$. Hence

$$\int_{S^3} (\psi \circ \varphi)^*\omega = \int_{S^3} \varphi^* \circ \psi^*\omega = \int_{S^3} \varphi^* 0 = 0.$$

Thus $\boxed{\int_{S^3} (\psi \circ \varphi)^*\omega = 0}$. □

Topology Exam

Summer 2014

Problem 6.5.1

Let M be a smooth manifold, let $x_0, x_1 \in M$, and let $\alpha, \beta : [0,1] \to M$ be smooth paths such that $x_0 = \alpha(0) = \beta(0)$ and $x_1 = \alpha(1) = \beta(1)$. We say that α is *smoothly path homotopic* to β if there exists a smooth map $h : [0,1] \times [0,1] \to M$ satisfying the conditions

- **For all $s \in [0,1]$, we have $h(s,0) = \alpha(s)$ and $h(s,1) = \beta(s)$.**
- **For all $t \in [0,1]$, we have $h(0,t) = x_0$ and $h(1,t) = x_1$.**

Let $\omega \in \Omega^1(M)$ be a closed smooth 1-form on M. Show that if α is smoothly path homotopic to β, then

$$\int_\alpha \omega = \int_\beta \omega.$$

Notes and Comments

Proof. Since h is a smooth map $[0,1]^2 \to M$, h is a smooth 2-cube on M. Then the boundary of h is the 1-chain

$$\partial h = \alpha + c_{x_1} - \beta - c_{x_0}$$

where $c_z : [0,1] \to M$ is the constant function $c_z(t) = z$.

Since ω is closed, $d\omega = 0$. Hence, by the parametrized Stokes' theorem,

$$0 = \int_h 0 = \int_h d\omega = \int_{\partial h} \omega = \int_\alpha \omega + \int_{c_{x_1}} \omega - \int_\beta \omega - \int_{c_{x_0}} \omega = \int_\alpha \omega - \int_\beta \omega.$$

Thus $0 = \int_\alpha \omega - \int_\beta \omega$ and so the result follows. $\square$

Problem 6.5.2

Let S be the surface

$$S = \left\{(x,y,z) \in \mathbb{R}^3 \mid x - yz + z^3 = 0\right\} \subseteq \mathbb{R}^3.$$

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $(x,y,z) \mapsto (x,y)$. Let H be the collection of points $p \in S$ such that $\pi|_S : S \to \mathbb{R}^2$ is *not* a local diffeomorphism in a neighborhood of p. Show that H is a smooth curve in $\mathbb{R}^3$ and determine a parametrization for it.

Notes and Comments

Proof. Coordinates on S are given by $\varphi : S \to \mathbb{R}^2$ where $\varphi(x,y,z) = (y,z)$.[6]

By the Inverse Function Theorem, $p \in H$ is equivalent to the pushforward $(\pi|_S)_{*,p}$ not being invertible. In coordinates, we have $\pi_S \circ \varphi^{-1}(y,z) = (yz - z^3, y)$ and we can easily[7] compute

$$(\pi_S \circ \varphi^{-1})_{*,(y,z)} = \begin{pmatrix} z & y - 3z^2 \\ 1 & 0 \end{pmatrix}.$$

[6]Just note that the x-coordinate is smoothly determined by y and z.

[7]... when not under qual pressure. Otherwise, all bets are off.

This matrix has full rank unless $y = 3z^2$. Hence $(x, y, z) \in H$ if and only if $y = 3z^2$ and $x = yz - z^3 = 2z^3$. That is

$$H = \left\{(2z^3, 3z^2, z) \mid z \in \mathbb{R}\right\}.$$

Thus we can parametrize H by $\boxed{h(z) = (2z^3, 3z^2, z)}$.

Without the parametrization, we can still tell that H is a smooth curve. By the Implicit Function Theorem, restricting to H, the pushforward has rank 1. So the codimension is 1 and that means that H is a 1-manifold in S. $\square$

Problem 6.5.3

Let M be a smooth n-manifold with smooth atlas of charts $\mathscr{A}$. Suppose that for all charts (x, U) and (y, V) in $\mathscr{A}$ with $U \cap V \neq \varnothing$, the change of charts map

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$$

has derivative

$$D(y \circ x^{-1})(x(p)) : \mathbb{R}^n \to \mathbb{R}^n$$

with positive determinant for every $p \in U \cap V$. Show that there is a nowhere vanishing smooth n-form $\omega \in \Omega^n(M)$.

Notes and Comments

Proof. For a chart (x, U) define

$$\nu_x = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

where $x = (x^1, x^2, \ldots, x^n)$. Let $\{\varphi_x\}_{(x,U)\in\mathscr{A}}$ be a partition of unity subordinate to $\mathscr{A}$. Then we can "glue" the local n-forms together using the partition of unity:

$$\nu = \sum_{(x,U)\in\mathscr{A}} \nu_x.$$

Then $\boxed{\nu \in \Omega^n(M)}$.

For charts (x, U) and (y, V) with $U \cap V \neq \varnothing$, we have $D(y \circ x^{-1})(x(p)) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map for $p \in U \cap V$. Thus, on $U \cap V$, we have

$$\nu_y = dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n = \det D(y \circ x^{-1})(dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n) = \det D(y \circ x^{-1})\nu_x$$

by top-degree pullback.[8] By assumption, $\det D(y \circ x^{-1}) > 0$ and so, in each chart, ν is nowhere vanishing. That is, $\boxed{\nu \text{ is nowhere vanishing on } M}$. $\square$

[8] It is, perhaps, improper to talk about the derivative/pushforward without referencing a point of the manifold. However, the physical and mental space saved by *not* referencing a point is well worth this disregard for details.

Problem 6.5.4

Let X and Y be topological spaces. Let $X \sqcup Y$ denote the disjoint union of X and Y, endowed with the coproduct topology, and let $i_X, i_Y : X, Y \to X \sqcup Y$ be the natural inclusion maps.

For any homology theory satisfying the Eilenberg–Steenrod axioms, prove that the induced maps

$$i_{X*} : H_q(X) \to H_q(X \sqcup Y) \text{ and } i_{Y*} : H_q(Y) \to H_q(X \sqcup Y)$$

induce an isomorphism

$$H_q(X) \oplus H_q(Y) \xrightarrow{\sim} H_q(X \sqcup Y)$$

for each $q \geq 0$.

Notes and Comments

Proof. Consider the pair $(X \sqcup Y, X)$. Then we have an associated long exact sequence

$$\cdots \longrightarrow H_q(X) \xrightarrow{i_{X*}} H_q(X \sqcup Y) \longrightarrow H_q(X \sqcup Y, X) \xrightarrow{\partial} H_{q-1}(X) \xrightarrow{i_*} \cdots$$

We claim this breaks up into short exact sequences.

Pick a point $x_0 \in X$. Define $r : X \sqcup Y \to X$ by $\begin{cases} r|_X = \mathrm{Id}_X \\ r|_Y = c_{x_0} \end{cases}$. Then $r \circ i_X = \mathrm{Id}_X$ and so r is a retraction of i_X. By functoriality, this means that

$$r_* \circ i_{X*} = (r \circ i_X)_* = \mathrm{Id}_{X*}\,.$$

Thus $\boxed{i_{X*} \text{ is injective}}$.

By exactness of the long exact sequence, $\operatorname{im} \partial = \ker i_{X*} = 0$. That is, $\boxed{\partial = 0}$. Thus

$$0 \longrightarrow H_q(X) \underset{r_*}{\overset{i_{X*}}{\rightleftarrows}} H_q(X \sqcup Y) \longrightarrow H_q(X \sqcup Y, X) \longrightarrow 0$$

is a split short exact sequence. Hence $\boxed{H_q(X \sqcup Y) \cong H_q(X) \oplus H_q(X \sqcup Y, X)}$.

Consider $(Y, \varnothing) = (X \sqcup Y \setminus X, X \setminus X) \overset{i_Y}{\hookrightarrow} (X \sqcup Y, X)$.

Since X is clopen in $X \sqcup Y$,

$$X = \mathring{X} = \overline{X}$$

and so by excision[9], i_Y induces an isomorphism $i_{Y*} : H_q(Y) \to H_q(X \sqcup Y, X)$. That is, we have $\boxed{H_q(Y) \cong H_q(X \sqcup Y, X)}$.

Thus $\boxed{H_q(X \sqcup Y) \cong H_q(X) \oplus H_q(Y)}$ via i_{X*} and i_{Y*} as desired. □

[9]The excision axiom states: Let $X \supseteq A \supseteq B$. If $\overline{B} \subseteq \mathring{A}$, then $(X \setminus B, A \setminus B) \hookrightarrow (X, A)$ induces isomorphism on $H_\bullet$.

Problem 6.5.5

Let $f : \mathbb{S}^2 \to \mathbb{T}^2$ be any continuous map from the 2-sphere $\mathbb{S}^2$ to the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Show that f is homotopic to a constant map.

Notes and Comments

Proof. Assume $f : \mathbb{S}^2 \to \mathbb{T}^2$ is continuous. Note that the universal cover of $\mathbb{T}^2$ is $\mathbb{R}^2$. Since $\mathbb{S}^2$ is path connected, f lifts to the universal cover of $\mathbb{T}^2$ by the Lifting Lemma. That is, there is a continuous map $\widetilde{f} : \mathbb{S}^2 \to \mathbb{R}^2$ such that

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{S}^2 \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & \mathbb{T}^2 \end{tikzcd}$$

commutes.[10]

Since $\pi_1(\mathbb{R}^2) = 0$, $\widetilde{f}$ is nullhomotopic. That is, there is a homotopy of $\widetilde{f}$ and a constant map $c : \mathbb{S}^2 \to \mathbb{R}^2$. Then $p \circ h$ is a homotopy of $p \circ \widetilde{f} = f$ and $p \circ c$, which is still a constant map. Hence f is nullhomotopic as desired. $\square$

Problem 6.5.6

See the Fall 2012 exam (problem 6).

Notes and Comments

Proof. The statement and solution for this problem are (effectively) identical. $\square$

[10] For complete precision, we should choose base points $x \in \mathbb{S}^2$, $f(x) \in \mathbb{T}^2$, and $e \in p^{-1}(f(x)) \subseteq \mathbb{R}^2$. ... Eh.

Topology Exam

Fall 2014

Problem 6.6.1

Let $C : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by $C(v, w) = v \times w$, the usual vector cross product. Determine the critical points of C. Conclude that for $0 \neq u \in \mathbb{R}^3$, the set $\{(v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid v \times w = u\}$ is a smooth manifold. If $\{e_1, e_2, e_3\}$ denotes the standard basis for $\mathbb{R}^3$, determine a basis of the tangent space $T_{(e_1,e_2)}(C^{-1}(e_3))$ as a vector subspace of $T_{(e_1,e_2)}(\mathbb{R}^3 \times \mathbb{R}^3) \cong \mathbb{R}^6$.

Notes and Comments

Proof. For vectors $v, w \in \mathbb{R}^3$, the cross product is given by $C(v, w) = \begin{pmatrix} v_2w_3 - v_3w_2 \\ -v_1w_3 + v_3w_1 \\ v_1w_2 - v_2w_1 \end{pmatrix}$. Thus the pushforward of C is

$$C_{*,(v,w)} = \begin{pmatrix} 0 & w_3 & -w_2 & 0 & -v_3 & v_2 \\ -w_3 & 0 & w_1 & v_3 & 0 & -v_1 \\ w_2 & -w_1 & 0 & -v_2 & v_1 & 0 \end{pmatrix}.$$

The critical points of C correspond to the pairs of vectors such that $\operatorname{rank} C_{*,(v,w)} < 3$.

Step 1: We claim that $C^{-1}(0) = \{\text{critical points of } C\}$.

Recall that $v \times w = 0$ is equivalent to $v = \lambda w$ for some $\lambda \in \mathbb{R}$. So we get one containment easily:

($\subseteq$): Assume $v = \lambda w$. Then

$$C_{*,(v,w)} = \begin{pmatrix} 0 & w_3 & -w_2 & 0 & -\lambda w_3 & \lambda w_2 \\ -w_3 & 0 & w_1 & \lambda w_3 & 0 & -\lambda w_1 \\ w_2 & -w_1 & 0 & -\lambda w_2 & \lambda w_1 & 0 \end{pmatrix}.$$

Now the submatrix $M = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}$ has determinant

$$\det M = -w_3(-w_3 \cdot 0 - (w_2w_1)) + (-w_2)(w_3w_1 - w_2(0)) = 0.$$

If $\lambda = 0$, switching in any other column won't help the determinant because we would have a column of all 0's. If $\lambda \neq 0$, switching in another column will multiply the determinant by λ or two columns will be linear combinations of one another. In all cases, we get that the determinant is 0. Thus every 3×3 submatrix of $C_{*,(v,w)}$ has determinant 0. That is, $\operatorname{rank} C_{*,(v,w)} < 3$. So (v, w) is a critical point of C.

($\supseteq$): Assume (v, w) is a critical point of C. We will show $v \times w = \begin{pmatrix} v_2w_3 - v_3w_2 \\ -v_1w_3 + v_3w_1 \\ v_1w_2 - v_2w_1 \end{pmatrix} = 0$. Consider the following submatrices of $C_{*,(v,w)}$:

$$A_1 = \begin{pmatrix} 0 & -v_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & v_1 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & w_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & -w_1 & 0 \end{pmatrix}$$

Then, since (v,w) is a critical point,

$$0 = \det A_1 = w_2(v_1w_3 - v_3w_1) \text{ and } 0 = \det A_2 = v_2(v_1w_3 - v_3w_1).$$

Thus $\boxed{v_1w_3 = v_3w_1}$ or $\boxed{v_2, w_2 = 0}$. If the latter holds,

$$C_{*,(v,w)} = \begin{pmatrix} 0 & w_3 & 0 & 0 & -v_3 & 0 \\ -w_3 & 0 & w_1 & v_3 & 0 & -v_1 \\ 0 & -w_1 & 0 & 0 & v_1 & 0 \end{pmatrix}.$$

Now consider

$$A_3 = \begin{pmatrix} w_3 & 0 & -v_3 \\ 0 & w_1 & 0 \\ -w_1 & 0 & v_1 \end{pmatrix} \text{ and } A_4 = \begin{pmatrix} w_3 & 0 & -v_3 \\ 0 & -v_1 & 0 \\ -w_1 & 0 & v_1 \end{pmatrix}.$$

Then

$$0 = \det A_3 = w_1(v_1w_3 - v_3w_1) \text{ and } 0 = \det A_4 = v_1(v_3w_1 - v_1w_3).$$

Thus $\boxed{v_1w_3 = v_3w_1}$ or $\boxed{w_1, v_1 = 0}$. In either case, $v \times w = 0$.

Going back to our original choices, we must consider the case where $v_1w_3 = v_3w_1$. By appropriate choices of other submatrices, we get:

$$(\boxed{v_2w_3 = v_3w_2} \text{ or } \boxed{v_1, w_1 = 0}) \text{ and } (\boxed{v_1w_2 = v_2w_1} \text{ or } \boxed{v_3, w_3 = 0}).$$

Analogous arguments show that we obtain the first equality in each pair regardless. Hence $v \times w = 0$.

Thus we have shown that the critical points are precisely $C^{-1}(0)$.

Step 2: For $u \neq 0$, we know by Step 1 that $C|_{C^{-1}(u)}$ has full rank. By the Implicit Function Theorem, $C^{-1}(u)$ is a 3-manifold (since $6 - \operatorname{rank} C_{*,(v,w)} = 6 - 3 = 3$).

Step 3: Consider a tangent vector $\alpha'(0) \in T_{(e_1,e_2)}(C^{-1}(e_3))$. That is, $\alpha : (-\varepsilon, \varepsilon) \to C^{-1}(e_3)$ is a smooth curve so that $\alpha(0) = (e_1, e_2)$. As a map into $\mathbb{R}^6$, we have that

$$\alpha(t) = (v_1(t), v_2(t), v_3(t), w_1(t), w_2(t), w_3(t))$$

where

$$\boxed{v_1(0) = w_2(0) = 1} \text{ and } \boxed{v_2(0) = v_3(0) = w_1(0) = w_3(0) = 0}.$$

Moreover, since $\alpha(t) \in C^{-1}(e_3)$, we know that

$$\boxed{v_1(t)w_3(t) = v_3(t)w_1(t)}, \boxed{v_2(t)w_3(t) = v_3(t)w_2(t)}, \text{ and } \boxed{v_1(t)w_2(t) - v_2(t)w_1(t) = 1}.$$

Taking the derivatives of these expressions at $t = 0$, we obtain (in order):

$$\boxed{w_3'(0) = 0}, \boxed{v_3'(0) = 0}, \boxed{w_2'(0) = -v_1'(0)}.$$

Thus $\alpha'(0) = (v_1'(0), v_2'(0), 0, w_1'(0), -v_1'(0), 0)$. Hence a basis for $T_{(e_1,e_2)}(C^{-1}(e_3))$ is

$$\boxed{\mathscr{B} = \{e_1 - e_5, e_2, e_4\}}.$$

$\square$

Problem 6.6.2

Let $p : Y \to X$ be a covering map. Let Z be any connected space and let $f : Z \to X$ be a continuous map. Suppose that $f_1 : Z \to Y$ and $f_2 : Z \to Y$ are continuous lifts of f (i.e., $p \circ f_i = f$ for $i = 1, 2$) that agree at some point $z_0 \in Z$. Show that $f_1 = f_2$ on all of Z.

Proof. See the solution to problem 5 on the Fall 2012 exam (6.2.5). □

Notes and Comments

Problem 6.6.3

Consider the circle $\mathbb{S}^1$ with its usual CW-structure with a single 0-cell e^0 and a single 1-cell e^1. Let X be the space obtained from $\mathbb{S}^1$ by attaching 2-cells e_1^2 and e_2^2 by maps of degree 2 and 3, respectively. Compute the homology groups of X.

Notes and Comments

Proof. The cellular chain groups W_q of X are

$$\cdots \longrightarrow 0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0 \quad .$$

That is $W_0 = \mathbb{Z}[e^0]$, $W_1 = \mathbb{Z}[e^1]$, and $W_2 = \mathbb{Z}[e_1^2, e_2^2]$. With respect to these bases, the boundary maps have matrix representations

$$[\partial_0] = \begin{pmatrix} 0 \end{pmatrix}, [\partial_1] = \begin{pmatrix} 0 \end{pmatrix}, \text{ and } [\partial_2] = \begin{pmatrix} 2 & 3 \end{pmatrix}$$

where the representation of ∂_2 uses the assumptions on how e_1^2 and e_2^2 were attached. The Smith normal form of $[\partial_2]$ is $\begin{pmatrix} 1 & 0 \end{pmatrix}$ and thus we have:

$$\begin{aligned} H_0(X) &= \frac{\ker \partial_0}{\operatorname{im} \partial_1} \cong \mathbb{Z} \\ H_1(X) &= \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\mathbb{Z}}{\mathbb{Z}} \cong 0 \\ H_2(X) &= \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \frac{\mathbb{Z}}{0} \cong \mathbb{Z}. \end{aligned}$$

Hence

$$\boxed{H_q(X) = \begin{cases} \mathbb{Z} & \text{if } q = 0, 2 \\ 0 & \text{else} \end{cases}}.$$

□

Problem 6.6.4

Let $f \in \mathbb{R}[x, y, z]$ be a homogeneous quadratic polynomial with real coefficients. Let $\mathbb{D}^3 = \{x \in \mathbb{R}^3 \mid \|x\| \le 1\}$ be the unit disk and $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ the unit sphere in Euclidean space $\mathbb{R}^3$. Let ν be the volume form on $\mathbb{S}^2$, where $\mathbb{S}^2$ is given the orientation induced from the standard orientation of $\mathbb{D}^3$. Prove that

$$\int_{\mathbb{D}^3} \Delta f = 2 \int_{\mathbb{S}^2} f \cdot \nu$$

where Δf is the Laplacian $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$. [Hint: Write Δf as a divergence.]

Notes and Comments

Proof. Observe that $\boxed{2f = x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z}}(*)$ since f is a homogeneous quadratic polynomial (for instance, one copy of cxy appears in $x\dfrac{\partial f}{\partial x}$ and the other in $y\dfrac{\partial f}{\partial y}$). Also note that $\nu_{\mathbb{D}^3} = dx \wedge dy \wedge dz$ is the standard orientation on $\mathbb{D}^3$ and $N = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$ is an outward pointing normal vector field (so $\nu = N \lrcorner \nu_{\mathbb{D}^3}$).

Define the vector field $V = \dfrac{\partial f}{\partial x}\dfrac{\partial}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{\partial}{\partial y} + \dfrac{\partial f}{\partial z}\dfrac{\partial}{\partial z}$. Then we have

$$\langle V, N\rangle = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} \overset{(*)}{=} 2f.$$

That is, $\boxed{\langle V, N\rangle = 2f}(**)$.

Using the dual basis expansion and omitting the gruesome computations, we obtain[11]

$$\begin{aligned} V \lrcorner \nu_{\mathbb{D}^3} &= dx(V)\ dy \wedge dz + (-1)^1 dy(V)\ dx \wedge dz + (-1)^2 dz(V)\ dx \wedge dy \\ &= \frac{\partial f}{\partial x}\ dy \wedge dz - \frac{\partial f}{\partial y}\ dx \wedge dz + \frac{\partial f}{\partial z}\ dx \wedge dy \end{aligned}$$

By definition, $\operatorname{div} V\ \nu_{\mathbb{D}^3} = d(V \lrcorner \nu_{\mathbb{D}^3})$ and so we obtain[12]

$$\operatorname{div} V\ \nu_{\mathbb{D}^3} = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)\nu_{\mathbb{D}^3} = \Delta f\ \nu_{\mathbb{D}^3}.$$

That is, $\boxed{\operatorname{div} V = \Delta f}$. Hence by the Divergence Theorem (†),

$$\int_{\mathbb{D}^3} \Delta f\ \nu_{\mathbb{D}^3} = \int_{\mathbb{D}^3} \operatorname{div} V\ \nu_{\mathbb{D}^3} \overset{(\dagger)}{=} \int_{\mathbb{S}^2} \langle V, N\rangle\ \nu \overset{(**)}{=} \int_{\mathbb{S}^2} 2f\ \nu = 2\int_{\mathbb{S}^2} f\ \nu.$$

Thus we have obtained the desired equality. □

Problem 6.6.5

Let $\mathbb{RP}^n$ denote real projective n-space. Show that if $n > 0$ is even, then every continuous map $f : \mathbb{RP}^n \to \mathbb{RP}^n$ has a fixed point.

Notes and Comments

[11] The intermediate step shown has been left to give the reader the idea that the jump should be obvious. In reality, writing down every detail would be tedious, time consuming, and unenlightening.

[12] This time, it's worse.

Proof. Since $\mathbb{RP}^n = \dfrac{\mathbb{S}^n}{p \sim -p}$, $\mathbb{S}^n$ is the 2-fold cover of $\mathbb{RP}^n$ with covering map p. So f lifts to a map $\widetilde{f} : \mathbb{S}^n \to \mathbb{RP}^n$ and thus we have a wonderful commuting diagram

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, dashed, "g"] \arrow[d, "p"'] \arrow[dr, "\widetilde{f}"] & \mathbb{S}^n \arrow[d, "p"] \\ \mathbb{RP}^n \arrow[r, "f"'] & \mathbb{RP}^n \end{tikzcd} .$$

Since $\mathbb{S}^n$ is path connected, $\widetilde{f}$ lifts to the universal cover of $\mathbb{RP}^n$ (which is $\mathbb{S}^n$ since $\pi_1(S^n) = 0$ for $n > 1$) by the Lifting Lemma.[13] Call this map $g : \mathbb{S}^n \to \mathbb{S}^n$.

Assume that f has no fixed points. Then $f(\overline{p}) \neq \overline{p}$ for all $\overline{p} \in \mathbb{RP}^n$. Consequently, $\widetilde{f}(p) \neq \overline{p}$ for all $p \in \mathbb{S}^n$. Hence $\boxed{g(p) \neq p, -p}$ for all $p \in \mathbb{S}^n$.

As g is a continuous map between n-spheres, we can compute its Brouwer degree.

- Since $g(p) \neq p$ for all p, g is homotopic to the antipodal map A. As Brouwer degree is preserved under homotopy, $\deg g = \deg A = (-1)^{n+1}$.[14] Since n is even, $\boxed{\deg g = -1}$.

- Since $g(p) \neq -p$ for all p, g is homotopic to the identity map. Thus $\boxed{\deg g = 1}$.

The contradiction is stunningly apparent and so f must have a fixed point. $\square$

Problem 6.6.6

Let M be a smooth n-manifold whose smooth structure is defined by a maximal atlas $\mathscr{M}$ of charts (x, U), where $U \subseteq M$ is open and $x : U \to x(U) \subseteq \mathbb{R}^n$ is a homeomorphism of U with an open subset of $\mathbb{R}^n$. Suppose that there is a nowhere vanishing smooth n-form $\omega \in \Omega^n(M)$. Show that there is a subcollection $\mathscr{A}$ of $\mathscr{B}$ such that

- **The collection $\{U \mid (x, U) \in \mathscr{A}\}$ cover M.**

- **For any two overlapping charts $(x, U), (y, V) \in \mathscr{A}$, i.e., any two charts in $\mathscr{A}$ such that $U \cap V \neq \varnothing$, the derivative $D(y \circ x^{-1})(x(p)) : \mathbb{R}^n \to \mathbb{R}^n$ of the mapping $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ has positive determinant for every $p \in U \cap V$.**

Notes and Comments

Proof. On a chart (x, U), $\omega = f_x \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ where $f_x \neq 0$ since ω is nowhere vanishing. Since f_x is smooth (in particular, continuous), either $f_x > 0$ or $f_x < 0$.

Let $\mathscr{A} = \{(x, U) \in \mathscr{M} \mid f_x > 0\}$. We will show that $\mathscr{A}$ is the desired collection.

- ($\mathscr{A}$ is an atlas) Suppose that $f_x < 0$ for a chart (x, U). Since $\mathscr{M}$ is a maximal atlas, the chart (x', U) where
$$x' = (x^2, x^1, x^3, x^4, \ldots, x^n)$$

[13] Once again, we *should* specify base points. Hah!

[14] A quick proof of this follows from the fact that A is the composition of $n+1$ reflections.

is in $\mathscr{M}$. Then

$$\omega = f_{x'} dx^2 \wedge dx^1 \wedge dx^3 \wedge dx^4 \wedge \cdots \wedge dx^n = \underbrace{f_{x'}(-1)}_{f_x} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

As $f_x < 0$, $f_{x'} = -f_x > 0$. Hence $(x', U) \in \mathscr{A}$. As the domains of the charts cover M, this shows that the domains of the charts in $\mathscr{A}$ also cover M.

- ($\mathscr{A}$ is compatible) For overlapping charts (x, U) and (y, V), we have

$$\begin{aligned} f_x dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = \omega &= f_y \; dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n \\ &= f_y \cdot \det D(y \circ x^{-1}) \; dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n. \end{aligned}$$

  That is, $f_x = f_y \cdot \det D(y \circ x^{-1})$. Since $f_x, f_y > 0$, we have $\boxed{\det D(y \circ x^{-1}) > 0}$.

Thus we have the desired atlas. $\square$

Topology Exam

Summer 2015

Problem 6.7.1

Show that any map $\mathbb{S}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ is nullhomotopic. [You may use without proof that $\mathbb{S}^2$ is simply connected.]

Notes and Comments

Proof. See the solution to problem 5 on the Summer 2014 exam (6.5.5). □

Problem 6.7.2

Consider the subspace X of $\mathbb{R}^3$ defined by $X = A \cup B$ where

$$A = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$$

is the xy-plane and

$$B = \{(0, y, z) \mid y, z \in \mathbb{R}, z \geq 0\}$$

is the upper half of the yz-plane. Show that X is not a topological manifold. [Hint: Consider local homology.]

Notes and Comments

Proof. If a space Y is a topological n-manifold, its local homology groups must agree with those of $\mathbb{R}^n$.[15] That is, for $y \in Y$, $H_q(Y \mid y) = \begin{cases} \mathbb{Z} & q = n \\ 0 & \text{else} \end{cases}$. We show that this is not true for X.

Recall that $H_q(X \mid x) = H_q(X, X \setminus \{x\})$. Picking our special point to be $0 \in \mathbb{R}^3$, we want to consider the reduced homology[16] associated to the pair $(X, X \setminus \{0\})$. Then the associated long exact sequence is:

$$\cdots \longrightarrow \widetilde{H}_q(X \setminus \{0\}) \longrightarrow \widetilde{H}_q(X) \longrightarrow H_q(X \mid 0) \xrightarrow{\partial} \widetilde{H}_{q-1}(X \setminus \{0\}) \longrightarrow \cdots$$

The space X is rather wonderfully constructed. Via the straight-line homotopy, X is contractible and so its reduced homology vanishes: $\boxed{\widetilde{H}_q(X) = 0}$. On the other hand, $X \setminus \{0\}$ is homotopy equivalent to $S^1 \vee S^1$:

- Consider $Y = S^2 \cap X$. This space consists of 3 semicircles joined at the points $(0, \pm 1, 0)$. Then the map $r : X \setminus \{0\} \to Y$ given by $r(x) = \dfrac{x}{\|x\|}$ is a deformation retraction of X onto Y. That is, X and Y are homotopy equivalent.
- Y and $S^1 \vee S^1$ are homotopy equivalent by contracting one semicircle to a point (thus joining the two endpoints of the remaining semicircles together).

[15] One can show that local homology on the whole space is the same as that in a neighborhood of a point. Since manifolds are locally $\mathbb{R}^n$, we can do a little dance and move on.

[16] One might reasonably ask: why reduced homology? Is the full-fat homology not good enough? No, it's not. Hold on!

Consequently, since S^1 is well-pointed and the reduced homology of the wedge product is the direct sum of reduced homologies, we have

$$\widetilde{H}_q(X \setminus 0) = \widetilde{H}_q(S^1 \vee S^1) = \widetilde{H}_q(S^1) \oplus \widetilde{H}_q(S^1) = \begin{cases} \mathbb{Z}^2 & q = 1 \\ 0 & \text{else} \end{cases}.$$

Hence our long exact sequence simplifies tremendously:

$$\cdots \longrightarrow \widetilde{H}_2(X) = 0 \longrightarrow H_2(X \mid 0) \xrightarrow{\partial} \mathbb{Z}^2 \longrightarrow 0 = \widetilde{H}_1(X) \longrightarrow \cdots$$

By exactness, ∂ is an isomorphism and so $\boxed{H_2(X \mid 0) \cong \mathbb{Z}^2}$. Thus the local homology of X does not agree with that of $\mathbb{R}^n$ and so X is not a manifold. □

Problem 6.7.3

See the Fall 2012 written exam (problem 6).

Notes and Comments

Proof. The statement and solution for this problem are (effectively) identical. □

Problem 6.7.4

Compute the Lie bracket $[V, W]$ of two vector fields

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \text{ and } W = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Notes and Comments

Proof. [17] By direct computation,

$$\begin{aligned} [V, W] &= \sum_{i,j=1}^{3} \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \\ &= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}. \end{aligned}$$

Alternatively, compute $V(Wf)$ and $W(Vf)$:[18]

$$V(Wf) = x\frac{\partial f}{\partial z} + xy\frac{\partial^2 f}{\partial y \partial z} - xz\frac{\partial^2 f}{\partial y^2} - y^2\frac{\partial^2 f}{\partial x \partial z} + yz\frac{\partial^2 f}{\partial x \partial y}$$

and

$$W(Vf) = yx\frac{\partial f}{\partial z \partial y} - y^2\frac{\partial^2 f}{\partial z \partial x} - zx\frac{\partial^2 f}{\partial y^2} + z\frac{\partial f}{\partial x} + zy\frac{\partial^2 f}{\partial y \partial x}.$$

[17] This shouldn't be called a proof. Who designed this thing anyway?

[18] Optionally, one can simply ignore the second-order partial derivatives because they necessarily cancel. This makes computation faster and less messy.

Then

$$[V,W] = VW - WV = x\frac{\partial}{\partial z} - z\frac{\partial f}{\partial x}.$$

$\square$

Problem 6.7.5

Let M^m be a compact m-dimensional manifold without boundary, with $m \geq 1$. Show that for all $k \geq 1$ there is no submersion $\phi : M \to \mathbb{R}^k$.

Notes and Comments

Proof. To the contrary, suppose that such a submersion ϕ exists for some $k \geq 1$. Then ϕ is a submersion between smooth manifolds without boundary and hence ϕ is an open map. Thus $\boxed{\phi(M)^{\text{open}} \subseteq \mathbb{R}^k}$.

Since M is compact and $\mathbb{R}^k$ is Hausdorff, $\boxed{\phi(M)^{\text{closed}} \subseteq \mathbb{R}^k}$. Thus $\phi(M)$ is clopen in $\mathbb{R}^k$. Since $\mathbb{R}^k$ is connected, this means that $\phi(M) = \mathbb{R}^k$. However, M is compact and $\mathbb{R}^k$ is not. Thus ϕ cannot be a submersion. $\square$

Problem 6.7.6

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map and $\omega = dx \wedge dy$ be a form on $\mathbb{R}^2$. Let

$$G = \{(x, y, f(x,y)) : (x,y) \in \mathbb{R}^2\} \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$$

be the graph of f and let

$$\pi_i : \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad i = 1, 2,$$

be the projections to the first and second factors. Let $W = \pi_1^*\omega - \pi_2^*\omega$. Show that $f^*\omega = \omega$ if and only if $W|_{TG} = 0$.

Notes and Comments

Proof. Firstly, by $W\,|_{TG}$ we mean $\iota_G^* W$ where $\iota_G : G \to \mathbb{R}^4$ is the inclusion map. Secondly, by chasing an element through, we can see that the following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, "\iota_G"] & \mathbb{R}^4 \arrow[d, "\pi_1 - \pi_2"] \\
\mathbb{R}^2 \arrow[u, "\mathrm{Id} \times f"] \arrow[r, "\mathrm{Id} - f"'] & \mathbb{R}^2
\end{tikzcd}$$

That is,

$$(\pi_1 - \pi_2) \circ \iota_G \circ (\mathrm{Id} \times f) = \mathrm{Id} - f.$$

Hence, using properties of the pullback,

$$\omega - f^*\omega = (\mathrm{Id} - f)^*\omega = ((\pi_1 - \pi_2) \circ \iota_G \circ (\mathrm{Id} \times f))^* \,\omega = (\mathrm{Id} \times f)^* \iota_G^*(\pi_1^*\omega - \pi_2^*\omega) = (\mathrm{Id} \times f)^* \left(W|_{TG}\right).$$

Since $\mathrm{Id} \times f : \mathbb{R}^2 \to G$ is a diffeomorphism, $(\mathrm{Id} \times f)^*$ is injective. Therefore $W|_{TG} \equiv 0$ if and only if $\omega = f^*\omega$. $\square$

Topology Exam

Fall 2015

Problem 6.8.1

Show that there is a map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$ that is not nullhomotopic. [Hint: Consider collapsing the 1-skeleton of a CW structure on the 2-torus to a point.]

Notes and Comments

Proof. Consider the torus $\mathbb{T}^2$ as the appropriate quotient of the unit square $[0,1]^2$ and the 2-sphere as the quotient obtained by gluing the entire boundary of $[0,1]^2$ to a single point $*$ via the continuous map $\widetilde{f} : [0,1]^2 \to \mathbb{S}^2$. Since the $\widetilde{f}$ is well-defined on equivalence classes of points in $\mathbb{T}^2$ (the only points being identified are $\partial[0,1]^2$ and they all map to $*$), we obtain a continuous surjective map $f : \mathbb{T}^2 \to \mathbb{S}^2$.

We claim that f induces an isomorphism $f_* : H_2(\mathbb{T}^2) \to H_2(\mathbb{S}^2)$. To see this, note that f is cellular (it maps the n-skeleton of $\mathbb{T}^2$ into the n-skeleton of $\mathbb{S}^2$). In particular, the 2-cell of the torus e^2 is mapped to the 2-cell of the sphere (identically on the interior). Since the boundary of the 2-cell is 0 in both cases, $f_*([e^2]) \neq 0$. Moreover, as the interiors are mapped identically, this must be the generator of $H_2(\mathbb{S}^2)$. That is f_* maps the generator of $H_2(\mathbb{T}^2)$ to the generator of $H_2(\mathbb{S}^2)$ and so f_* is an isomorphism on second homology.

As homotopy preserves homology, f cannot be nullhomotopic (the one-point space has trivial second homology). $\square$

Problem 6.8.2

Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a map such that $f(\mathbb{D}^n_+) \subseteq \mathbb{D}^n_+$ and $f(\mathbb{D}^n_-) \subseteq \mathbb{D}^n_-$, where $\mathbb{D}^n_\pm$ are the northern and southern hemispheres of $\mathbb{S}^n$. Show that $f(\mathbb{S}^{n-1}) \subseteq \mathbb{S}^{n-1}$, and $\deg(f) = \deg(f|_{\mathbb{S}^{n-1}})$.

Notes and Comments

Proof. Since the intersection $\mathbb{D}^n_+ \cap \mathbb{D}^n_- = \mathbb{S}^{n-1}$ and f maps a hemisphere into itself, the points of intersection must map to other points of the intersection. That is, $\boxed{f(\mathbb{S}^{n-1}) \subseteq \mathbb{S}^{n-1}}$. We show that f is homotopic to the suspension of a self-map on $\mathbb{S}^{n-1}$.

Consider $g = \Sigma\left(f|_{\mathbb{S}^{n-1}}\right)$. Since $\mathbb{D}^n_\pm$ is contractible, any two continuous maps into $\mathbb{D}^n_\pm$ are homotopic. As suspension preserves the northern and southern hemispheres, g restricts to self-maps of $\mathbb{D}^n_\pm$. As f does this as well, and f and g agree on $\mathbb{S}^{n-1}$, we obtain homotopies of $f|_{\mathbb{D}^n_\pm}$ and $g|_{\mathbb{D}^n_\pm}$ relative to $\mathbb{S}^{n-1}$ which we call $h_\pm$.

The map

$$h : \mathbb{S}^n \times [0,1] \to \mathbb{S}^n \text{ given by } h(p,t) = \begin{cases} h_+(p,t) & p \in \mathbb{D}^n_+ \\ h_-(p,t) & p \in \mathbb{D}^n_- \end{cases}$$

is a homotopy of f and g. That is, h is continuous by the Pasting Lemma ($h_\pm(,t)$ agree on $\mathbb{S}^{n-1}$).

Since g is the suspension of a map on $\mathbb{S}^{n-1}$, $\boxed{\deg g = \deg f|_{\mathbb{S}^{n-1}}}$. As we just showed that $f \simeq g$, and homotopy preserves degree, $\boxed{\deg f = \deg g = \deg f|_{\mathbb{S}^{n-1}}}$. $\square$

Problem 6.8.3

Let $p : Y \to X$ be a covering map. Let Z be any connected space and let $f : Z \to X$ be a continuous map. Suppose that $f_1 : Z \to Y$ and $f_2 : Z \to Y$ are continuous lifts of f (i.e., $p \circ f_i = f$ for $i = 1, 2$) that agree at some point $z_0 \in Z$. Show that $f_1 = f_2$ on all of Z.

Notes and Comments

Proof. See the solution to problem 5 on the Fall 2012 exam (6.2.5). □

Problem 6.8.4

You are given a smooth map of manifolds $\pi : M \to N$ such that every $x \in M$ is in the image of a local smooth section from a neighborhood $U_{\pi(x)}$ of $\pi(x)$ into a neighborhood of x. Prove that π is a submersion.

Notes and Comments

Proof. Let $p \in M$ and consider the smooth local section $\sigma : U_{\pi(p)} \to M$ such that $\sigma(\pi(p)) = p$. Since $\pi \circ \sigma = \mathrm{Id}_{U_{\pi(p)}}$, we have

$$\pi_{*,p} \circ \sigma_{*,\pi(p)} = \mathrm{Id}_{T_{\pi(p)}N} .$$

by functoriality. That is, $\pi_{*,p}$ is surjective. Since p was arbitrary, π_* is surjective and thus π is, by definition, a submersion. □

Problem 6.8.5

Show that the 2-sphere S^2 admits a continuous vector field with exactly one zero point.

Notes and Comments

Proof. Let $N = (0, 0, 1) \in \mathbb{R}^3$ and $\varphi : S^2 \setminus \{N\} \to \mathbb{R}^2$ be stereographic projection. Then $(\varphi, S^2 \setminus \{N\})$ is a chart on S^2.

Let (u, v) be the coordinates in $\mathbb{R}^2$ and define a smooth vector field V on $\mathbb{R}^2$ by $V_{(u,v)} = \dfrac{\partial}{\partial u}$. Then V doesn't vanish on $\mathbb{R}^2$ and hence $W = \varphi_*^{-1} V$ is smoothly defined and nonvanishing on $S^2 \setminus \{N\}$. We may continuously extend W to N since W is continuous on every punctured disk centered at N.

Since the 2-sphere doesn't admit a global nonvanishing vector field (by the Hairy Ball Theorem), it must be that the limit of $\varphi_*^{-1} V$ along all points going to N is the 0 vector.[19] Hence we may define $W_N = 0$. So we obtain a continuous vector field

$$W_p = \begin{cases} \varphi^{-1}_{*,\varphi^{-1}(p)} \dfrac{\partial}{\partial u} & p \neq N \\ 0 & p = N. \end{cases} .$$

Since W only vanishes at N, we have the desired vector field. □

[19] What's more, this is actually a smooth extension of W. However, to see that, we need to move into the coordinates on $\mathbb{R}^2$ for both stereographic projection maps and use the results there to make this conclusion. We doubt that was expected for this exam.

Problem 6.8.6

Differential forms.

(a) Let $\phi : S^1 \times S^1 \to \mathbb{R}^5$ be a smooth map defined on the 2-torus, where S^1 is the 1-sphere. Let ω be a closed 2-form on $\mathbb{R}^5$. Compute the integral $\int_{S^1 \times S^1} \phi^* \omega$.

(b) Let $\phi : M \to S^1 \times S^1$ and $\psi : S^1 \times S^1 \to M$ be two smooth maps, where M is a compact oriented 4-manifold. Let ω be a 4-form on M. Compute $\int_M (\psi \circ \phi)^* \omega$.

Notes and Comments

Proof. See the solution to problem 6 on the Fall 2013 written exam (6.4.6). The problems are visually distinct but the solutions are the same. □

Topology Exam

Summer 2016

Problem 6.9.1

Let M^m, N^m be smooth manifolds of the same dimension, and suppose that M is compact and N is connected. Show that if $f : M \to N$ is a submersion, it is a covering.

Notes and Comments

Proof. Since f is a submersion, it is an open map. So $f(M)$ is open in N. Also, since M is compact and f continuous, $f(M)$ is compact in N. As N is Hausdorff, $f(M)$ is closed in N. That is, $f(M)$ is clopen in a connected space; hence $f(M) = N$.

Knowing that f is surjective and open, it is enough to show that f is a local homeomorphism. However, this is done for us: as f is a submersion, its pushforward is surjective. As $\dim M = \dim N$, this means that f_* is an isomorphism at any point $p \in M$. That is, f is a local diffeomorphism (hence homeomorphism) and thus a covering map. □

Problem 6.9.2

Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and consider the map $r : S^n \to S^n$ defined by $r(x) = -x$. Show that r is orientation-reversing if and only if n is even.

Notes and Comments

Proof. One characterization of orientation is that it's a choice of non-vanishing n-form. We will compute the volume form ν_{S^n} of S^n using the induced orientation from D^{n+1}, the unit $(n+1)$-disk sitting inside $\mathbb{R}^{n+1}$.

We know that D^{n+1} has volume form $\nu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n+1}$. To compute the induced orientation of D^{n+1} on $\partial D^{n+1} = S^n$, we need a smooth (outward-pointing) vector field on D^{n+1}. Take $\boxed{N(x) = x}$.

Letting $\lrcorner$ denote the interior product,

$$\begin{aligned}\nu_{S^n} = N \lrcorner \nu &= \sum_{i=1}^{n+1} (-1)^{i-1} N(x^i)\ dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1} \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} x^i\ dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}.\end{aligned}$$

Then using our antipodal map r and the wonderful properties of the exterior derivative (d),

$$\begin{aligned}r^*\nu_{S^n} &= \sum_{i=1}^{n+1} (-1)^{i-1} r^*(x^i)\ dr^*x^1 \wedge \cdots \wedge dr^*x^{i-1} \wedge dr^*x^{i+1} \wedge \cdots \wedge dr^*x^{n+1} \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} (-x^i)\ d(-x^1) \wedge \cdots \wedge d(-x^{i-1}) \wedge d(-x^{i+1}) \wedge \cdots \wedge d(-x^{n+1}) \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} (-1)^{n+1} x^i\ dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1} \\ &= (-1)^{n+1} \nu_{S^n}.\end{aligned}$$

That is, $\boxed{r^*\nu_{S^n} = (-1)^{n+1}\nu_{S^{n-1}}}$. So r is orientation reversing if and only if $(-1)^{n+1} = -1$. That is, if and only if n is even. □

Problem 6.9.3

Let $(x_1, \ldots, x_n)$ be the standard coordinates on $\mathbb{R}^n$ and $(y_1, \ldots, y_{n+1})$ the standard coordinates on $\mathbb{R}^{n+1}$. Let $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the map given by

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sum_{i=1}^{n} x_i^2).$$

Compute the induced metric f^*g on $\mathbb{R}^n$ where $g = \sum_{j=1}^{n+1} dy_j^2$ is the standard metric on $\mathbb{R}^{n+1}$. Your answer should be expressed in the form $\sum_{i,j} g_{ij}dx_idx_j$ where the g_{ij} are smooth real-valued functions on $\mathbb{R}^n$.

Notes and Comments

Proof. When it comes to Riemannian metrics, it's enough to know the coefficient functions. That is, to compute f^*g, we need to know $g_{ij} = f^*g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$.[20] By definition of pullback, we have

$$f^*g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g\left(f_*\left(\frac{\partial}{\partial x_i}\right), f_*\left(\frac{\partial}{\partial x_j}\right)\right).$$

In order to compute these, however, we need to understand the pushforward of f.

At a point x, we can reasonably compute the Jacobian of f:

$$f_{*,x} = \left[\begin{array}{ccc} & I_n & \\ \hline 2x_1 & \cdots & 2x_n \end{array}\right]$$

Thus we have $\boxed{f_{*,x}\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} + 2x_i\frac{\partial}{\partial y_{n+1}}}$. Now, using the fact that g is the usual metric on $\mathbb{R}^{n+1}$ (i.e., it arises from the usual inner product), we have

$$\begin{aligned} f^*g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) &= g\left(\frac{\partial}{\partial y_i} + 2x_i\frac{\partial}{\partial y_{n+1}}, \frac{\partial}{\partial y_j} + 2x_j\frac{\partial}{\partial y_{n+1}}\right) \\ &= \delta_{ij} + (2x_i)(2x_j) \\ &= \delta_{ij} + 4x_ix_j \end{aligned}$$

[20] This may seem slightly inappropriate given all the other naming conventions around. However, by now, you've probably given in to the morass that is symbology in differential topology and so we will capitalize on that.

where δ_{ij} is the Kronecker delta function. Thus $\boxed{g_{ij} = \begin{cases} 4x_ix_j & i \neq j \\ 1 + 4x_i^2 & i = j \end{cases}}$. Hence

$$f^*g = \sum_{i \neq j} (4x_ix_j)dx_idx_j + \sum_{i=1}^{n} (1 + 4x_i^2)dx_i^2.$$

□

Problem 6.9.4

Show that if A is a deformation retraction of X and $x_0 \in A$, then the induced map $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism, where $i : A \to X$ is the inclusion map.

Notes and Comments

Proof. Since A is a deformation retract of X, there is a continuous map $r : X \to A$ such that $r|_A = \mathrm{Id}_A$ and $i \circ r \simeq_A \mathrm{Id}_X$.[21] By functoriality, since $x_0 \in A \subset X$, we get

$$\mathrm{Id}_{X*} = (i \circ r)_* = i_* \circ r_*$$

as maps on π_1 with base point x_0.[22] Thus $\boxed{i_* \text{ is surjective}}$.

Since $r \circ i = \mathrm{Id}_A$, we also obtain

$$\mathrm{Id}_{A*} = (r \circ i)_* = r_* \circ i_*.$$

That is, $\boxed{i_* \text{ is injective}}$; hence an isomorphism as desired. □

Problem 6.9.5

Let X be the topological space obtained from two copies of the 2-sphere S^2 and one copy of the circle $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ by identifying the point $1 \in S^1$ with the north pole of the first 2-sphere and the point $-1 \in S^1$ with the north pole of the second 2-sphere. Draw the universal cover of X.

Notes and Comments

Proof. The important things to realize in this problem are:

- S^2 is simply-connected and so its universal cover is itself.
- The universal cover of S^1 is $\mathbb{R}$, generally viewed as a helix $(\cos(2\pi z), \sin(2\pi z), z)$.

Now we can draw the universal cover of $(S^2 \vee_{-1} S^1) \vee_1 S^2$ (here $-1, 1 \in S^1$ are the points to which the north poles of respective spheres are attached) by joining a sphere to the helix at $z = k$ and another one at $z = k + \frac{1}{2}$ for $k \in \mathbb{Z}$. □

[21] The extra decoration means that the homotopy fixes A. This is important to apply the π_1 functor: we need the base point, which sits in A, fixed.

[22] That is, $r_* : \pi_1(X, x_0) \to \pi_1(A, r(x_0))$ and $i_* : \pi_1(A, x_0) \to \pi_1(A, i(x_0))$. Since $r(x_0) = i(x_0) = x_0$, this is actually the map we claimed.

Problem 6.9.6

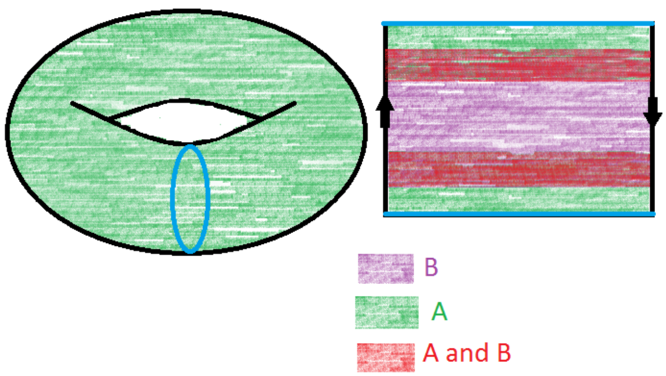
Use the Mayer–Vietoris sequence to compute all the homology groups of the space X obtained from the torus $S^1 \times S^1$ by attaching a Möbius band M via the homeomorphism from the boundary circle of M to the circle $S^1 \times \{x_0\} \subset S^1 \times S^1$.

Notes and Comments

Proof. There are two major tricks to working through this problem with the Mayer–Vietoris sequence. First, we need to determine our sets A and B. Second, and most unfortunately, we need to understand the maps used in the sequence for the homology to work out.

To solve our first problem, we consider the realization of M as a square. Since the boundary of M is being glued to the meridian of the torus, we're going to take N to be a connected neighborhood of the boundary in the quotient X. Let $A = (S^1 \times S^1) \cup N$ and B be the open Möbius band contained in M.

Now A and B are open sets in X. Pictorially, this corresponds to taking the torus and "a little bit" of the square for the Möbius band to be A.[23]



Hence $A \simeq S^1 \times S^1$ and $B \simeq M \simeq S^1$. That is, A is really just a torus and B is, after a deformation retraction, just S^1. Now $A \cap B$ is also important for Mayer–Vietoris and, in this case, N is just a cylinder and hence homotopy equivalent to S^1.

Now A and B are open sets which cover X and so Mayer–Vietoris applies to the pair. That is, we have a long exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & H_2(X) \arrow[r] & H_1(A\cap B) \arrow[r] & H_1(A)\oplus H_1(B) \arrow[r] & H_1(X) \arrow[dllll] \\
H_0(A\cap B) \arrow[r] & H_0(A)\oplus H_0(B) \arrow[r] & H_0(X) \arrow[r] & 0
\end{tikzcd}$$

Since we "know" what A, B, and $A \cap B$ are, we can simplify this considerably. That is, we know $H_q(A) =$

[23] Really we need to specify more about the neighborhood N but we'll let the pictures fill in for us.

$$\begin{cases} \mathbb{Z} & q = 0,2 \\ \mathbb{Z} \oplus \mathbb{Z} & q = 1 \\ 0 & \text{else} \end{cases} \text{ and } H_q(B) = H_q(A \cap B) = \begin{cases} \mathbb{Z} & q = 0,1 \\ 0 & \text{else} \end{cases}.$$ Thus our long exact sequence looks like:[24]

$$\begin{tikzcd} \ldots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & H_2(X) \arrow[r] & \mathbb{Z} \arrow[r] & (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} \arrow[r] & H_1(X) \arrow[dllll] \\ & & \mathbb{Z} \arrow[r, "f"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "g"'] & H_0(X) \arrow[r] & 0 & \end{tikzcd}$$

Note that we don't have to go further out in the sequence because $H_q(X)$ is surrounded by 0 for $q > 2$. Hence, by exactness, $\boxed{H_q(X) = 0 \text{ for } q > 2}$. Also, X is path connected and so $\boxed{H_0(X) = \mathbb{Z}}$.

By exactness, g is surjective. Also, $\mathbb{Z} = \ker g = \operatorname{im} f$. That is, f is injective and the previous map must be the 0 map. So we can simplify our sequence more (while naming new maps):

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_2(X) \longrightarrow \mathbb{Z} \xrightarrow{\alpha} (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} \longrightarrow H_1(X) \xrightarrow{0} 0$$

Let's focus on the map $\alpha : \mathbb{Z} \to \mathbb{Z}^3$ coming from the map $H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ and think about what's going on. The homology class is coming from the loop γ around the meridian $S^1 \times \{x_0\}$ and the map sends the meridian to its copy in A and the copy in B.

In A, γ is exactly the map that goes around the meridian once and longitude not at all. In B, the loop goes around the core of the Möbius band twice. Thus the map α is given by $\alpha(1) = (0, 1, 2)$ (with basis corresponding to: longitude, meridian, core of the Möbius band). Thus α is injective and so, by exactness, the previous map is the 0 map. So our sequence splits into short exact sequences:

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(X) \xrightarrow{0} 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}^3 \longrightarrow H_1(X) \longrightarrow 0$$

By exactness, $\boxed{H_2(X) \cong \mathbb{Z}}$. Also by exactness,

$$H_1(X) \cong \frac{\mathbb{Z}^3}{\operatorname{im} \alpha}.$$

Now the image of α is $\mathbb{Z}$. Hence

$$H_1(X) \cong \frac{\mathbb{Z}^3}{\mathbb{Z}} \cong \mathbb{Z}^2.$$

Thus we have

$$\boxed{H_q(X) = \begin{cases} \mathbb{Z} & q = 0,2 \\ \mathbb{Z}^2 & q = 1 \\ 0 & \text{else} \end{cases}}.$$

□

[24] Look! More terms appeared! It's stunning how much space these sequences can take up.

Topology Exam

Fall 2016

Problem 6.10.1

Show that the group $\mathrm{SL}(n, \mathbb{R})$ consisting of all $n \times n$ matrices with determinant 1 is a smooth manifold.

Notes and Comments

Proof. There are a number of different approaches to this problem. We consider two(ish):

Option 1: Note that $\mathrm{GL}(n, \mathbb{R})$ is a smooth manifold since it's an open subset of $\mathbb{R}^{n^2}$ (which is the quintessential smooth manifold).[25] Moreover, $\mathrm{GL}(n, \mathbb{R})$ is actually a Lie group.

Now $\mathrm{SL}(n, \mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$ is a subgroup. Perhaps more importantly, $\mathrm{SL}(n, \mathbb{R})$ is a closed subset. That is,by determinant considerations, it contains all its limit points. Thus $\mathrm{SL}(n, \mathbb{R})$ is a closed subgroup of a Lie group; hence a Lie group in its own right.

Option 2(a): We may instead show that $\mathrm{SL}(n, \mathbb{R}) = \det^{-1}(\{1\})$ is an embedded submanifold using the Implicit Function Theorem. Indeed, consider the determinant map. Since det is a map to $\mathbb{R}$ (which is one-dimensional), it's enough to show that $\det|_{\mathrm{SL}(n,\mathbb{R})*}$ has nonzero rank.[26]

Indeed, for $A \in \mathrm{SL}(n, \mathbb{R})$, the directional derivative agrees with the push-forward and so

$$\det_{*,A}(A) = \frac{d}{dt}\Big|_{t=0} \det(A + tA) = \frac{d}{dt}\Big|_{t=0} (1+t)^n \det(A) = \frac{d}{dt}\Big|_{t=0} (1+t)^n = n.$$

That is, $\det_{*,A} \neq 0$ for all $A \in \mathrm{SL}(n, \mathbb{R})$. Hence by the Implicit Function Theorem, $\mathrm{SL}(n, \mathbb{R})$ is an embedded submanifold of $\mathrm{GL}(n, \mathbb{R})$.

Option 2(b): If you already know that the critical points of the determinant map are contained in the non-invertible matrices (see Fall 2012 #2), then you can immediately make the conclusion using the Implicit Function Theorem. □

Problem 6.10.2

Let $p \in M$ be a point in a smooth manifold M and let $\mathcal{F}_p$ be the subspace of $C^\infty(M)$ consisting of all smooth functions that vanish at p. Let $\mathcal{F}_p^2 \subset \mathcal{F}_p$ be the subspace spanned by functions of the form fg for $f, g \in \mathcal{F}_p$. Define a map $\Phi : \mathcal{F}_p \to T_p^*M$ by setting

$$\Phi(f) = df_p.$$

Show that the restriction of Φ to $\mathcal{F}_p^2$ is zero and that Φ descends to an isomorphism $\mathcal{F}_p/\mathcal{F}_p^2 \to T_p^*M$ of vector spaces.

Hint: You can use the following fact without a proof. Let ϕ be a local chart centered at p (i.e., ϕ is a chart defined on a neighborhood of p such that $\phi(p) = 0$); then $f \in \mathcal{F}_p$ if and only if $(f \circ \phi^{-1})(x_1, \ldots, x_n) = \sum_{i=1}^n x_i f_i(x_1, \ldots, x_n)$ for some smooth functions f_i.

Notes and Comments

[25] To check this, use the determinant map.

[26] This will further show that $\dim \mathrm{SL}(n, \mathbb{R}) = n^2 - 1$.

Proof. Let $h \in \mathcal{F}_p^2$. We want to show $\Phi(h) = 0$. By definition, $h = \sum_{i=1}^{k} f_i g_i$ for $f_i, g_i \in \mathcal{F}_p$. Using properties of the differential d and the definition of $\mathcal{F}_p$, we have

$$\Phi(h) = d\left(\sum_{i=1}^{k} f_i g_i\right)_p = \sum_{i=1}^{k} d(f_i g_i)_p = \sum_{i=1}^{k}\left(\underbrace{f_i(p)}_{0}\, d(g_i)_p + \underbrace{g_i(p)}_{0}\, d(f_i)_p\right) = 0.$$

Now suppose that $h \in \ker \Phi$. In a local chart ϕ centered at p, we have

$$h \circ \phi^{-1} = \sum_{i=1}^{n} x^i h_i$$

by the hint (where x^i is the projection onto the ith component). Since $h \in \ker \Phi$ and $\phi(p) = 0$, we have $d(h \circ \phi^{-1})_0 = 0$. That is

$$0 = d(h \circ \phi^{-1})_0 = \sum_{i=1}^{n}(x^i(0) d(h_i)_0 + h_i(0) dx_0^i) = \sum_{i=1}^{n} h_i(0) dx_0^i$$

since $x^i(0) = 0$. Since the dx_0^i are linearly independent (indeed, form a basis for T_p^*M), $\boxed{h_i(0) = 0}$ for all i. That is, $h_i \circ \phi \in \mathcal{F}_p$. Thus we have

$$h = \sum_{i=1}^{n}(x^i \circ \phi)(h_i \circ \phi)$$

in coordinates centered at p. As $x^i \circ \phi$ and $h_i \circ \phi$ are locally defined, they can be extended to smooth maps f_i and g_i (respectively) defined on all of M using smooth bump functions. Hence $f_i, g_i \in \mathcal{F}_p$ and $h = \sum_i f_i g_i$. That is, $h \in \mathcal{F}_p^2$.

This shows that $\boxed{\ker \Phi = \mathcal{F}_p^2}$. Finally, Φ descends to an injective linear map

$$\overline{\Phi} : \mathcal{F}_p / \mathcal{F}_p^2 \to T_p^* M.$$

As T_p^*M is finite-dimensional, we conclude that $\overline{\Phi}$ is an isomorphism as desired. □

Problem 6.10.3

Let $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$ be the torus defined by

$$T^2 = \left\{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 = 1, z^2 + t^2 = 1\right\}$$

with the orientation determined by its product structure, where each circle factor is oriented as the boundary of the unit disk. Compute $\int_{T^2} z dx \wedge dt$.

Notes and Comments

Proof. Notice that $\omega = zdx \wedge dt$ is an exact 2-form. Indeed, take the 1-form $\eta = zxdt$ and note that

$$d\eta = zdx \wedge dt + xdz \wedge dt = \omega$$

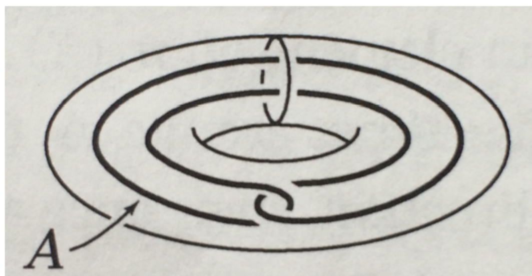
since $dz \wedge dt$ is a 2-form on the 1-dimensional manifold S^1 (and so the second term vanishes). Thus, by Stokes' Theorem,

$$\int_{T^2} \omega = \int_{\partial T^2} \eta = \int_{\varnothing} \eta = 0.$$

Hence the integral is 0. □

Problem 6.10.4

Let A be the curve inside the solid torus $S^1 \times D^2$ pictured in the figure below. Show that there is no retraction of the solid torus onto A.



Notes and Comments

Proof. Suppose we have a retraction r of $S^1 \times D^2$ onto A. Then the inclusion map $i : A \to S^1 \times D^2$ induces an injective map i_* on the fundamental groups of these spaces. Since A is simply a circle, $\pi_1(A, x) = \mathbb{Z}$ for any $x \in A$. The generating loop α traverses A exactly as shown in the image. However, $i(\alpha)$ is contractible in $S^1 \times D^2$: the "hooked" ends can be homotoped past each other in the solid torus and consequently the loop can be pulled to the base point x. Thus $i_*([\alpha])$ is the trivial loop. $\lightning$ Hence no such retraction can exist. □

Problem 6.10.5

Let X be the quotient space of S^2 under the identification $x \sim -x$ for x in the *equator* of S^2. Compute the homology groups $H_i(X;\mathbb{Z})$ for all i.

Notes and Comments

Proof. Consider the CW-complex structure for X by taking the CW-complex structure for S^2 consisting of 2 vertices, 2 edges, and 2 faces and then identifying the edges (hence the vertices). Then the cellular chain groups W_i of X are

$$\cdots \longrightarrow 0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

where $W_0 = \mathbb{Z}[v]$, $W_1 = \mathbb{Z}[e]$, and $W_2 = \mathbb{Z}[f_1, f_2]$. We assume that, imagining f_1 sitting above the plane and f_2 below, both are oriented counterclockwise (as viewed from above). Furthermore, for convenience, assume e is oriented counterclockwise as well. The matrix representations of the boundary maps are

$$[\partial_0] = \begin{pmatrix}0\end{pmatrix}, [\partial_1] = \begin{pmatrix}0\end{pmatrix}, \text{ and } [\partial_2] = \begin{pmatrix}2 & 2\end{pmatrix}.$$

Hence the Smith normal form of $[\partial_2]$ is $\begin{pmatrix} 2 & 0 \end{pmatrix}$ and so the homology groups are

$$\begin{aligned} H_0(X) &= \frac{\ker \partial_0}{\operatorname{im} \partial_1} \cong \mathbb{Z} \\ H_1(X) &= \frac{\ker \partial_1}{\operatorname{im} \partial_2} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \\ H_2(X) &= \frac{\ker \partial_2}{\operatorname{im} \partial_3} \cong \mathbb{Z} \end{aligned}$$

That is, we have $H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}/2\mathbb{Z} & i = 1 \\ 0 & \text{else} \end{cases}$. □

Problem 6.10.6

Show that for finite CW-complexes X and Y, the Euler characteristic χ satisfies

$$\chi(X \times Y) = \chi(X) \times \chi(Y).$$

Notes and Comments

Proof. This problem is a matter of writing down the relevant definitions. Let x_i be the number of i-cells of X and y_i the number of i-cells for Y. Then

$$\chi(X) = \sum_{i=0}^{k} (-1)^i x_i \text{ and } \chi(Y) = \sum_{j=0}^{k} (-1)^j y_j$$

where k is the largest dimension of cell in either X or Y (k exists because they're both finite complexes). Then the product $X \times Y$ has a CW-structure consisting of d_m m-dimensional cells for $m \le 2k$.

Note that the product $e^i \times e^j$ of an i-cell and a j-cell is a $(i+j)$-cell. Hence

$$d_m = \sum_{i+j=m} x_i y_j$$

and so

$$\begin{aligned} \chi(X \times Y) = \sum_{m=0}^{2k} (-1)^m d_m &= \sum_{m=0}^{2k} \sum_{i+j=m} (-1)^i x_i \cdot (-1)^j y_j \\ &= \left(\sum_{i=0}^{k} (-1)^i x_i \right) \left(\sum_{j=0}^{k} (-1)^j y_j \right) = \chi(X)\chi(Y). \end{aligned}$$

Thus we have the desired equality. □

Topology Exam

Summer 2017

Problem 6.11.1

Prove that a nonempty smooth manifold M of dimension m cannot be diffeomorphic to an n-dimensional manifold N, unless $m = n$.

Notes and Comments

Proof. Suppose $F : M \to N$ is a diffeomorphism and $p \in M$. Then F_{p*} is an isomorphism between T_pM and $T_{F(p)}N$. Since finite dimensional vector spaces are isomorphic if and only if they have the same dimension,

$$m = \dim M = \dim T_pM = \dim T_{F(p)}N = \dim N = n.$$

Thus $\boxed{m = n}$ as desired. □

Problem 6.11.2

Which of the following manifolds are parallelizable? (Recall that an m-dimensional manifold M is parallelizable if it admits m vector fields whose values at every point of M are linearly independent.) Explain your answers.

(a) S^2**;**

(b) S^2 **minus a point;**

(c) S^2 **minus two points;**

(d) $\mathrm{SO}(n, \mathbb{R})$**;**

(e) the Klein bottle;

(f) an oriented compact surface of genus 4 with no boundary.

Notes and Comments

Proof of (a). S^2 is not parallelizable by the Hairy Ball Theorem. Any vector field on S^2 must vanish at some point and so any possible collection of vector fields fail to be linearly independent at such points. □

Proof of (b). S^2 minus a point is diffeomorphic to $\mathbb{R}^2$ which is parallelizable by the coordinate vector fields $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$. □

Proof of (c). S^2 minus two points is diffeomorphic to $\mathbb{R}^2$ minus the origin, which is parallelizable by taking the frame consisting of $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ and the outward-pointing radial vector field. □

Proof of (d). $\mathrm{SO}(n, \mathbb{R})$ is a Lie group and is thus parallelizable. □

Proof of (e). Every parallelizable manifold is necessarily orientable. Thus the Klein bottle is not parallelizable. □

Proof of (f). An oriented compact surface without boundary is parallelizable if and only if its Euler characteristic is 0. Since a genus 4 surface has Euler characteristic -6, this surface is not parallelizable. □

Problem 6.11.3

Let M_1, M_2 be oriented compact k-dimensional manifolds with no boundary, and let $\phi_i : M_i \to N$, $i = 1, 2$, be smooth maps of manifolds. Assume, moreover, that there exists a $(k+1)$-dimensional oriented compact manifold Σ whose oriented boundary is $\partial\Sigma = M_1 \sqcup -M_2$, and a smooth map $F : \Sigma \to N$ such that $F|_{\partial\Sigma} = \phi_1 \sqcup \phi_2$. Let ω be a closed k-form on N. Prove that $\displaystyle\int_{M_1} \phi_1^*\omega = \int_{M_2} \phi_2^*\omega$.

Notes and Comments

Proof. Since $\partial\Sigma = M_1 \sqcup -M_2$ and $F|_{\partial\Sigma} = \phi_1 \sqcup \phi_2$,

$$\int_{\partial\Sigma} F^*\omega = \int_{M_1 \sqcup -M_2} F^*\omega = \int_{M_1} F|_{M_1}^*\omega - \int_{M_2} F|_{M_2}^*\omega = \int_{M_1} \phi_1^*\omega - \int_{M_2} \phi_2^*\omega.$$

So it suffices to show that $\displaystyle\int_{\partial\Sigma} F^*\omega = 0$. By Stokes' theorem, we see that $\displaystyle\int_{\partial\Sigma} F^*\omega = \int_\Sigma dF^*\omega$. As d commutes with everything and ω is closed,

$$\int_\Sigma dF^*\omega = \int_\Sigma F^*d\omega = \int_\Sigma F^*0 = 0.$$

Thus $\displaystyle\int_{M_1} \phi_1^*\omega = \int_{M_2} \phi_2^*\omega$, as desired. □

Problem 6.11.4

$\mathbb{R}P^n$ has a standard CW structure with a single k-cell for $k = 0, 1, \dots, n$. Prove that there is no retract from this CW complex onto its 1-skeleton.

Notes and Comments

Proof. Unstated in this problem is that $n > 1$. Since $n > 1$, recall that $H_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$.[27]

Let X_1 denote the 1-skeleton of $\mathbb{R}P^n$, i the inclusion of X_1 into $\mathbb{R}P^n$, and suppose $r : \mathbb{R}P^n \to X_1$ is a retraction. Then $H_1(X_1) = \mathbb{Z}$ since X_1 is topologically a circle. By the functoriality of homology, $r_* \circ i_* = \mathrm{Id}_{X_1*}$. In particular, we have

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[dr, "\mathrm{Id}_{X_1*}"'] & \mathbb{Z}/2\mathbb{Z} \arrow[d, "r_*"] \\ & \mathbb{Z} \end{tikzcd}$$

. That is, $\mathbb{Z}$ factors through $\mathbb{Z}/2\mathbb{Z}$. ↯ Hence no such retraction can exist. □

[27] This can be derived using cellular homology for anyone interested in a cute computation.

Problem 6.11.5

Use a Mayer–Vietoris sequence to prove the isomorphism of reduced homology groups $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ for $n \geq 1, k \geq 1$.

Notes and Comments

Proof. For a Mayer–Vietoris sequence, we need to decompose the larger space cleverly. Since we're working with $X = S^n = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : |x| = 1\}$, the natural decomposition $X = A \cup B$ where $A = \left\{x \in S^n : x_{n+1} > -\frac{1}{17}\right\}$ is the upper hemisphere (and a tiny bit more) and $B = \left\{x \in S^n : x_{n+1} < \frac{1}{17}\right\}$ is the lower hemisphere (and a tiny bit more).[28] Notice that A, B are both contractible because they're homeomorphic to n-balls and so $\boxed{\widetilde{H}_k(A) = \widetilde{H}_k(B) = 0}$.

The intersection $A \cap B$ is a band $S^{n-1} \times (-\frac{1}{17}, \frac{1}{17})$ and thus deformation retracts on S^{n-1}. Hence $A \cap B$ can be replaced by S^{n-1} for the purposes of homology.[29]

The Mayer–Vietoris sequence for reduced homology associated to (X, A, B) is given by:

$$\ldots \longrightarrow \widetilde{H}_k(A) \oplus \widetilde{H}_k(B) \longrightarrow \widetilde{H}_k(X) \xrightarrow{\partial} \widetilde{H}_{k-1}(A \cap B) \longrightarrow \widetilde{H}_{k-1}(A) \oplus \widetilde{H}_{k-1}(B) \longrightarrow \ldots \ .$$

Using the helpful notes above, this simplifies to

$$\ldots \longrightarrow 0 \longrightarrow \widetilde{H}_k(S^n) \xrightarrow{\partial} \widetilde{H}_{k-1}(S^{n-1}) \longrightarrow 0 \longrightarrow \ldots$$

By exactness of the long exact sequence, ∂ is an isomorphism $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ as desired. □

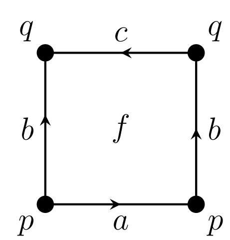
Problem 6.11.6

Identify S^1 with the complex unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $X = S^1 \times [0,1]/\sim$ be the quotient space obtained from the cylinder $S^1 \times [0,1]$ by identifying points $(z, 0) \sim (iz, 0) \sim (-z, 0) \sim (-iz, 0)$ for any $z \in S^1$, and likewise $(z, 1) \sim (iz, 1) \sim (-z, 1) \sim (-iz, 1)$ for any $z \in S^1$ at the other boundary component. Here $i = \sqrt{-1}$.

Compute the homology groups $H_n(X; \mathbb{Z})$.

Notes and Comments

Proof. Multiplication by i is a rotation through angle $\dfrac{\pi}{2}$ about the origin. The equivalence relation on $S^1 \times [0,1]$ is identifying points on the top (resp. bottom) copy of S^1 by this action of i. We can give X a CW structure as follows:



[28]We chose $\frac{1}{17}$ rather than ε for $\varepsilon < 1$ because we had trouble choosing a fun number. Later editors of this text can be more clever.

[29]Formally, the deformation retract induces an isomorphism on homology.

where each 0-cell and 1-cell is attached by degree 1 maps as depicted, and the 2-cell f is attached by $bc^4b^{-1}a^4$. Note that f is attached this way to account for the rotational action of i on the top and bottom circles.

Now the cellular chain complexes for this CW structure are: $0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{0} 0$.

In the bases $\{a, b, c\}$ for 1-cells and $\{p, q\}$ for 0-cells, $[\partial_1] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Thus $\boxed{\ker \partial_1 \cong \mathbb{Z}^2}$ and $\boxed{\operatorname{im} \partial_1 \cong \mathbb{Z}}$.

Computing the Smith normal form for ∂_2 gives

$$[\partial_2] = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \sim \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}.$$

Hence $\boxed{\ker \partial_2 = 0}$ and $\boxed{\operatorname{im} \partial_2 = 4\mathbb{Z}}$.

Thus the homology groups of X are

$$H_0(X; \mathbb{Z}) \cong \frac{\mathbb{Z}^2}{\mathbb{Z}} \cong \mathbb{Z}, \qquad H_1(X; \mathbb{Z}) \cong \frac{\mathbb{Z}^2}{4\mathbb{Z}} \cong \mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z}), \qquad H_2(X; \mathbb{Z}) \cong 0.$$

□

7

Applied

Applied Exam

Summer 2017

Problem 7.1.1

Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function having a fixed point x^* and the fixed point iteration be defined by $x_{k+1} = g(x_k)$, $k = 1, \ldots$.

(a) Sketch a proof of the result:

If $|g'(x^*)| < 1$, then the iteration is locally convergent if $|g'(x^*)| > 1$ the fixed point iteration diverges for any starting point other than x^*.

(b) Use your results from part (a) to determine the rate of convergence.

(c) Use Taylor's theorem to deduce the condition under which the iteration converges quadratically.

(d) Is Newton's method for finding a zero of a smooth function $f : \mathbb{R} \to \mathbb{R}$ an example of such a fixed point iteration scheme? If so, what is the function g in this case? If not, then explain why not.

(e) For the following two functions determine whether the fixed point iteration converges locally to either of the real roots of $x^4 = 1/16$:

i. $g_1(x) = x + x^4 - 1/16$

ii. $g_2(x) = 1 + x - 16x^4$.

Notes and Comments

Proof. (a) The error after the $n + 1$ iterations is $\epsilon_{n+1} = |x_{n+1} - x^*| = |g(x_n) - g(x^*)|$ since x^* is a fixed point. Taylor expanding $g(x_n)$ about x^*, we have $\epsilon_{n+1} = \left|g'(x^*)\epsilon_n + \frac{g''(x^*)}{2}\epsilon_n^2 + \cdots\right|$. Throwing out higher order terms, we have $\epsilon_{n+1} \approx |g'(x^*)|\epsilon_n$.

For local convergence, we need $\frac{\epsilon_{n+1}}{\epsilon_n} < 1$, which we see is satisfied precisely if $|g'(x^*)| < 1$. For any initial guess that isn't x^*, if $|g'(x^*)| > 1$ then the error will grow with each step.

(b) In general, following the Taylor expansion procedure, we get an equation for the error $\epsilon_{n+1} = |c \cdot \epsilon_n^\alpha + \cdots|$. The rate of convergence is α, the lowest power of ϵ_n. E.g. if $\alpha = 1$ we say that the convergence is linear.

(c) Using (a), we see that if $g'(x^*) = 0$ then convergence is at least quadratic, i.e. $\alpha \geq 2$.

(d) Yes, Newton's method is a fixed point iteration scheme where $g(x) = x - \frac{f(x)}{f'(x)}$.

(e) The two real roots are $\pm\frac{1}{2}$.

i. We compute $g_1'(x) = 1 + 4x^3$ and find $\left|g_1'\left(\frac{1}{2}\right)\right| = 1.5 > 1$ and $\left|g_1'\left(-\frac{1}{2}\right)\right| = 0.5 < 1$. Hence the iteration is divergent for $\frac{1}{2}$ and convergent (for good initial guesses) for $-\frac{1}{2}$.

ii. We compute $g_2'(x) = 1 - 64x^3$ and find $\left|g_2'\left(\frac{1}{2}\right)\right| = 7 > 1$ and $\left|g_2'\left(\frac{1}{2}\right)\right| = 9 > 1$. Hence the iteration is divergent for both $\pm\frac{1}{2}$.

$\square$

Problem 7.1.2

Let A_n be the 2×2 matrix given by

$$A_n = \begin{bmatrix} 1 & 2 \\ 2 & 4 + 1/n^2 \end{bmatrix}$$

(a) Find A_n^{-1} and the condition number of A_n. (Use the one norm to calculate the condition number).

(b) Let $n = 100$. Use the Gaussian elimination without pivoting to solve $A_{100} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = b$ using 5 significant figures at all stages of the calculation when

$$b = (1, 2 - 1/n^2)^T.$$

(c) Repeat part (b) using 2 significant figures in the calculation.

(d) Explain the answers in parts (b) and (c).

Notes and Comments

Proof. (a) First, compute

$$A_n^{-1} = \begin{bmatrix} 4n^2 + 1 & -2n^2 \\ -2n^2 & n^2 \end{bmatrix}.$$

Now, compute the condition number

$$\kappa(A_n) = ||A_n||_1 ||A_n^{-1}||_1 = (6 + 1/n^2)(6n^2 + 1) = 36n^2 + 12 + 1/n^2.$$

(b) We want to solve the system

$$\begin{bmatrix} 1 & 2 \\ 2 & 4.0001 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1.9999 \end{bmatrix}.$$

Using Gaussian elimination with 5 significant digits, we find the solution $x = 3$, $y = -1$.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4.0001 & 1.9999 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & .0001 & -.0001 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

(c) Using 2 significant digits, we find infinitely many solutions of the form $x + 2y = 1$.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4.0001 & 1.9999 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

(d) In the case of an ill-conditioned system ($\kappa(A_{100}) \approx 360000 \gg 1$), using too few significant digits will result in round-off error and incorrect solutions.

□

Problem 7.1.3

Consider the well-posed initial value problem $u_t = f(t, u)$, $t > 0$, with $u(0) = u_0$. Suppose we use the following scheme to solve this IVP, where h is the (fixed) time step:

$$y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4}(4f_{n+1} - f_n + 3f_{n-1}),$$

(a) Find the order of accuracy of this scheme and the leading term of truncation error.

(b) Does this scheme satisfy the *root condition*? Explain. Is this scheme *zero stable*? Explain.

(c) Define absolute stability. What is the difference between zero and absolute stability? Derive the equation for the absolute stability region for this scheme. Your solution should be an explicit expression for time step h. You do not have to solve it.

Notes and Comments

Proof. (a) To find the order of accuracy, we first compute the step error

$$|u_{n+1} - y_{n+1}| = |u_n + u_n'h + \frac{u_n''}{2}h^2 + \frac{u_n'''}{6}h^3 + \cdots - \frac{1}{2}(y_n + y_{n-1}) - \frac{h}{4}(4f_{n+1} - f_n + 3f_{n-1})|.$$

Taylor expand y_{n-1}, f_{n+1}, and f_{n-1}, group terms in powers of h, and cancel to find

$$\begin{aligned} |u_{n+1} - y_{n+1}| &= |u_n + u_n'h + \frac{u_n''}{2}h^2 + \frac{u_n'''}{6}h^3 + \cdots - \frac{1}{2}(y_n + y_{n-1}) - \frac{h}{4}(4f_{n+1} - f_n + 3f_{n-1})| \\ &= |u_n + u_n'h + \frac{u_n''}{2}h^2 + \frac{u_n'''}{6}h^3 + \cdots - \frac{1}{2}(y_n + u_n - u_n'h + \frac{u_n''}{2}h^2 - \frac{u_n'''}{6}h^3 + \cdots) \\ &\quad - \frac{h}{4}(4(u_n' + u_n''h + \frac{u_n'''}{2}h^2 + \cdots) - u_n' + 3(u_n' - u_n''h + \frac{u_n'''}{2}h^2 + \cdots))| \\ &= |-\frac{5}{8}u_n'''h^3 + \cdots|. \end{aligned}$$

The leading step error term is $-\frac{5}{8}u_n'''h^3$, so this is a second order method.

(b) Yes, this scheme satisfies the root condition since the roots of $\rho(r) = r^2 - r/2 - 1/2$ are $r = -1/2, 1$ which both have modulus less than or equal to 1. Yes, this scheme is zero stable, since satisfying the root condition is equivalent to zero stability.

(c) A scheme is absolutely stable if all roots of $\Pi(r)$ have modulus less than or equal to 1. Conversely, a scheme is zero stable if it satisfies the root condition, i.e. all roots of $\rho(r)$ have modulus less than or equal to 1.

To get the region, first find the roots of $\Pi(r) = (1-h)r^2 + (-1/2 + h/4)r - 1/2 - 3h/4$, which we obtain using the quadratic formula

$$r = \frac{h - 2 \pm \sqrt{-47h^2 + 12h + 36}}{8(h-1)}.$$

The absolute stability region is defined by h such that $\left|\frac{h-2\pm\sqrt{-47h^2+12h+36}}{8(h-1)}\right| \leq 1$. As stated, we don't need to solve for the exact region.

□

Problem 7.1.4

Consider a random walk on the integers such that the transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q$ for all integers i ($0 < p < 1$, $p + q = 1$).

(a) Determine the n-step transition probability $p_{00}^{(n)}$.

(b) Find the generating function of $u_n = p_{00}^{(n)}$, i.e. $P(x) = \sum_{n=0}^{\infty} u_n x^n$.

(c) Determine the generating function of the recurrence time from state 0 to state 0.

(d) What is the probability of eventual return to the origin?

Notes and Comments

Proof. (a) Traveling from the origin to the origin requires an even number of total steps, as well as the same number of left and right steps. Hence we have the formula

$$u_n = p_{00}^{(n)} = \begin{cases} \binom{n}{n/2}(pq)^{n/2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

(b) Compute the generating function using part (a):

$$P(x) = \sum_{n=0}^{\infty} u_n x^n = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \binom{n}{n/2}(pq)^{n/2}x^n = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \binom{n}{n/2}(\sqrt{pq}x)^n = \frac{1}{\sqrt{1-4pqx^2}}.$$

(c) Recall the relation $P(x) = \frac{1}{1-F(x)}$, where $F(x) = \sum_{n=0}^{\infty} f_0^{(n)} x^n$ is the generating function for the recurrence time from state 0 to state 0. Compute:

$$F(x) = \frac{P(x) - 1}{P(x)} = \frac{\frac{1}{\sqrt{1-4pqx^2}} - 1}{\frac{1}{\sqrt{1-4pqx^2}}} = 1 - \sqrt{1-4pqx^2}.$$

(d) The probability of eventual return to the origin is $f_0 = \sum_{n=1}^{\infty} f_0^{(n)} = F(1)$. Compute:

$$F(1) = 1 - \sqrt{1 - 4pq}.$$

□

Problem 7.1.5

Consider a critical homogeneous birth-and-death process (birth rate λ = death rate μ), starting with one single individual.

(a) Write down the Kolmogorov backward equation for the probability $p_{1j}(t)$ that the population size transitions from 1 to j at time t.

(b) Using (a), derive the backward recursive equation for the probability generating function $P(x,t)$ for the population size $X(t)$ distribution at t. And solve $P(x,t)$.

(c) Find the probability $p_0(t)$ that the population becomes extinct by time t.

(d) What is the mean extinction time?

(e) What is the probability that the population size *ever* reaches n?

Notes and Comments

Proof. (a) Let $\lambda = \mu = r$. The backward Kolmogorov equation is

$$\frac{dp_{1j}(t)}{dt} = -2rp_{1j}(t) + rp_{2j}(t) + rp_{0j}(t).$$

(b) We know the general formula for the probability generating function,

$$P(x,t) = \sum_{j=0}^{\infty} p_{1j}(t)x^j.$$

Using (a), we find the backward recursive equation for $P(x,t)$

$$\begin{aligned}\frac{d\sum_{j=0}^{\infty} p_{1j}(t)}{dt} &= -2r\sum_{j=0}^{\infty} p_{1j}(t) + rp_{2j}(t) + r\sum_{j=0}^{\infty} p_{0j}(t)\\ \implies \frac{dP(x,t)}{dt} &= -2rP(x,t) + rP(x,t)^2 + r\end{aligned}$$

with initial condition $P(x,0) = x$. Solve using separation of variables:

$$\begin{aligned}\frac{dP}{dt} &= -2rP + rP^2 + r\\ \implies \frac{dP}{(P-1)^2} &= rdt\\ \implies \int \frac{dP}{(P-1)^2} &= \int rdt\\ \implies \frac{1}{1-P} &= rt + c\\ \implies P(x,t) &= \frac{rt + c - 1}{rt + c}.\end{aligned}$$

Now use the initial condition $P(x,0) = x$:

$$x = P(x,0) = \frac{c-1}{c}$$
$$\implies c = \frac{1}{1-x}.$$

Finally arrive at the solution (and simplify a bit):

$$P(x,t) = \frac{rt + \frac{1}{1-x} - 1}{rt + \frac{1}{1-x}} = \frac{rt(1-x)+x}{rt(1-x)+1}.$$

(c) The probability of going extinct before time t is

$$p_0(t) = P(0,t) = \frac{rt}{rt+1}.$$

(d) Part (c) gives us the cumulative function for probability, but we need the density to find the mean extinction time. So taking the derivative, we find that the mean extinction time is ∞ since

$$\int_0^\infty t\frac{r}{(rt+1)^2}dt \to \infty.$$

(e) View $n \geq 1$ as an absorbing state. Starting with one single individual in a critical birth-death process, recall that the probability of fixation is $\frac{1}{n}$.

$\square$

Problem 7.1.6

Let X be a nonnegative integer-valued random variable with probability generating function $f(x) = \sum_{n=0}^\infty a_n x^n$. After observing X, then conduct X binomial trials with probability p of success. Let Y denote the resulting number of successes.

(a) Determine the generating function of Y.

(b) Determine the generating function of X given that $Y = X$.

(c) Suppose that for every p $(0 < p < 1)$ the probability generating functions in (a) and (b) coincide. Prove that the distribution of X is Poisson, $f(x) = e^{\lambda(x-1)}$ for some $\lambda > 0$.

Notes and Comments

Proof. (a) Compute

$$\begin{aligned}
g(x) &= \sum_{k=0}^{\infty} P(Y=k)x^k \\
&=^{(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P(Y=k|X=j)P(X=j)x^k \\
&=^{(2)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \binom{j}{k} p^k(1-p)^{j-k}a_j x^k \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{j}{k} (px)^k(1-p)^{j-k}a_j \\
&= \sum_{j=0}^{\infty}(1-p+px)^j a_j \\
&= f(1-p+px)
\end{aligned}$$

where (1) is by conditional expectation and (2) is by the formula for binomial probability.

(b) Using Bayes Theorem, we have

$$\begin{aligned}
h(x) &= \sum_{k=0}^{\infty} P(X=k|Y=X)x^k \\
&= \sum_{k=0}^{\infty} \frac{P(Y=X|X=k)P(X=k)}{\sum_{j=0}^{\infty} P(Y=X|X=j)P(X=j)}x^k \\
&= \sum_{k=0}^{\infty} \frac{a_k p^k}{\sum_{j=0}^{\infty} a_j p^j}x^k \\
&= \sum_{k=0}^{\infty} \frac{a_k (px)^k}{f(p)} \\
&= \frac{f(px)}{f(p)}.
\end{aligned}$$

(c) Suppose $f(1-p+px) = \frac{f(px)}{f(p)}$ for all p with $0 < p < 1$. Then we also have

$$\begin{aligned}
\frac{\partial}{\partial x}\bigg|_{x=1} f(1-p+px) &= \frac{\partial}{\partial x}\bigg|_{x=1} \frac{f(px)}{f(p)} \\
\implies f'(1) &= \frac{f'(p)}{f(p)}.
\end{aligned}$$

Viewing this as an ODE in p with condition $f(1) = 1$, we find $f(p) = e^{f'(1)(p-1)}$. Notice that this is the same form as $f(x) = e^{\lambda(x-1)}$ with $\lambda = f'(1)$, hence the distribution of X is Poisson.

$\square$

Written Qualifying Exam: Commentary

8

Algebra Commentary

Algebra Commentary

Summer 2012

Problem 8.1.1

Problem 1 Notes

Solution

(a) The amount of time devoted to cyclotmic polynomials in Math 111 varies greatly from year to year. However, this particular problem doesn't require any particularly specialized knowledge (just how to factor $x^n - 1$ over $\mathbb{Q}$).

Familiarity with the proof and applications of the Correspondence Theorem is recommended as this is a frequent qual topic.

(b) The key idea here is that it is much easier to work with coefficients in $\mathbb{Z}/5\mathbb{Z}$ since, as a field, the polynomial ring is a PID. The realization that $\mathbb{Z}[x]/\langle p \rangle \cong (\mathbb{Z}/p\mathbb{Z})[x]$ is a straightforward computation (if you are not already familiar with it). Similar to the note for part (a), the amount of time devoted to extensions of finite fields depends heavily on the instructor but the fact that there is a unique-up-to-isomorphism field of each prime power order is assumed knowledge.

Problem 8.1.2

Problem 2 Notes

Solution

The properties of splitting fields of $x^n - a$ over $\mathbb{Q}$ are another frequently occurring qual topic. Luckily, Lang devotes an entire section of his book to exactly these extensions. Knowing the matrix embedding of these extensions can also be very helpful if you need to determine explicit Galois group actions or fixed fields. The method presented here for showing that the intersection is trivial is a useful one that should be in your Galois toolkit.

The diagram presented in the problem is gratuitous.[1]

[1]But how could we not include a diagram one?

Problem 8.1.3

Problem 3 Notes

Solution

This is a version of the classic follow-your-nose problem. Anytime we want to define a map on a tensor object, we want to make our definition on elementary tensors (really pairs of elements from each factor) and extend by linearity. This problem is particularly nice because everything in sight is a vector space and so obtaining an isomorphism is simplified – simply take bases to bases.

Different faculty members prefer different "definitions" of naturality in the context of Math 101. The formal functorial definition was desired on this particular exam but some instructors may only want you to observe the fact the a basis is not needed to define the map.

Problem 8.1.4

Problem 4 Notes

Solution

One of the most important techniques from Math 101 is studying linear maps of vector spaces using the $k[x]$-module formulation. Knowing the various decomposition theorems and their relations to canonical forms is highly recommended.

Interpreting the problem in terms of basic linear algebra we see that part (a) is just asking for the construction of an eigenvector, part (b) is showing that eigenvectors corresponding to different eigenvalues are linearly independent, and part (c) is the implication that, if the geometric and algebraic multiplicities agree for each eigenvalue, the operator is diagonalizable.

Problem 8.1.5

Problem 5 Notes

Solution

Finding normal p-Sylow subgroups with the Sylow theorems usually relies on showing $n_p = 1$ using the modular equivalence class of n_p to determine possible values and then some case work with counting to rule out possibilities. Dummit and Foote has many examples of this process. Other key ingredients of this proof include the fact that unique p-Sylow subgroups are normal, subgroups whose index is the smallest prime in the group are normal, and that there is only one group of order 15 (also in Dummit and Foote).

The inner semi–direct product criterion also shows up in several Galois problems. A group G with subgroup H and normal subgroup N is a inner semi–direct product $G = N \rtimes H$ if $N \cap H = \{e\}$ and $G = NH$. Another neat way to show that $G = N \rtimes H$ is to exhibit a homomorhpism $\varphi : G \to H$ so that $\varphi|_H = \mathrm{Id}$ and $\varphi(N) = e$.

Problem 8.1.6

Problem 6 Notes

Solution

This is a neat problem that really only relies on the basic facts about multiplicativity in towers and properties of the minimal polynomial. Don't overthink it.

Algebra Commentary

Fall 2012

Problem 8.2.1

Problem 1 Notes

Solution

This is a very standard type of Galois problem and you should be able to fill in the details of the proof steps. A full treatment of this situation is given in Lang's *Algebra*. The polynomial factors over $\mathbb{Q}$ as $x^{15} - 8 = (x^5 - 2)(x^{10} + 2x^5 + 4)$. Compare with 4.1.2.

Problem 8.2.2

Problem 2 Notes

Solution

You should definitely have a proof of this result prepared for the written exam. This particular proof is our favorite but there are many others. Find one that works for you. Also, the tools that you have to approach this problem will vary with the course instructor. For example, many steps in this proof are simplified by implicit assumption of the Fundamental Lemma but having to prove that on the spot might not be ideal.

Problem 8.2.3

Problem 3 Notes

Solution

This problem usually feels mysterious the first time you see it on a homework assignment. Hopefully after having spent some time with the S_n embedding in the context of simple extensions, etc., it feels like familiar material. Bonus points[2] if you can write down some polynomials with $|G| = n!$.

Problem 8.2.4

Problem 4 Notes

Solution

This is a fairly straightforward application of the various Sylow theorems and standard counting arguments. Pieces of this result occur on many different exams, so knowing the structure of the proof is very important. In a pinch you may simply be able to claim that some part of this result is true[3] on your way to proving something else, but most faculty members will be expecting to see the proof.

One note about the way the proof is presented here: the casework is organized so that you see the individual counts separately, but this can obscure the global proof by contradiction. The bigger picture is that, if there is not a normal subgroup, there are too many group elements. It is this argument that is most commonly used in other specific cases, so it is good to recognize that structure.

The remainder of the proof relies on some simple results from group theory. Review your Dummit and Foote if the proofs of these results feel a little rusty.

[2] Not really...

[3] For example, the result that in a group of order pqr there is a normal Q or R subgroup.

Problem 8.2.5

Problem 5 Notes

Solution

Smith Normal Form is not always taught in 101 but it is a very useful tool, particularly in Algebraic Topology. It is important to remember that the final isomorphism is not necessarily the one that comes from the obvious choice of basis.

Problem 8.2.6

Problem 6 Notes

Solution

A common approach for proving facts about projective modules is: prove the result for free modules (which are well-behaved and have bases) and then prove it for projective modules as summands of free modules. That approach, plus the fact that tensor products distribute over direct sums, are the key components to this proof.

Algebra Commentary

Summer 2013

Problem 8.3.1

Problem 1 Notes

Solution

This problem is not "hard" but requires some some playing around. The key insight is that Sylow subgroups are conjugate, so we can use this to transform a statement about $q \in P_1 \cap P_2$ to one about $P \cap (\text{Something Else})$.

Problem 8.3.2

Problem 2 Notes

Solution

Part (a) seems obvious as stated and is really just careful definition checking. If the adjective "entire" does not appeal to you, there are other options. Part (b) is a little more complex but hopefully, if this type of problem shows us on your exam, you will have gotten some practice with this method on homework assignments. The key idea is to try to get a little more leverage out of the Zeroth Isomorphism Theorem than you usually need.

Part (c) is not really related to the first two parts, but is again just definition checking with just a little bit of the CRT tossed in for good measure. The likelihood of this type of problem appearing depends heavily on who taught the course.

Problem 8.3.3

Problem 3 Notes

Solution

Almost the entire content of this problem lies in parsing the problem statement. If you can identify that the first sentence of the proof is equivalent to the first sentence of the problem then you should be good to go. The rest is just some simple (if slightly tedious) algebra.

Problem 8.3.4

Problem 4 Notes

Solution

Galois basics. You should review the distinguished extension properties if your class used Lang as a textbook. It is probably also worth reviewing the "finite implies algebraic" argument.

Problem 8.3.5

Problem 5 Notes

Solution

Another follow-your-nose type naturality proof. Part (b) is interesting because the direct computation is necessary to discover the conditions. Don't be afraid to get your hands dirty...

Problem 8.3.6

Problem 6 Notes

Solution

The fact that this problem can be formulated in terms of cosets is quite interesting since it applies in the common situations where not all of our interesting subgroups are normal. It also makes for a good qual problem because the group theory portion is not difficult but does rely on understanding and not memorization of results. Part (b) is probably most useful/observed in the setting of algebraic number theory.

Algebra Commentary

Fall 2013

Problem 8.4.1

Problem 1 Notes

Solution

This is in some sense just a series of fairly straightforward computations and checking of your knowledge and comfort with basic-linear-algebra-as-module-theory. The level of detail that you would include for each step of this problem depends heavily on the focuses of your Math 101 class. Notice that you can save some work once you determine the (μ_T, χ_T) pairs by using the theorem that T is diagonalizable over $\mathbb{F}$ if and only if μ_T is square-free and splits.

Problem 8.4.2

Problem 2 Notes

Solution

Ring theory basics. Parts (b) and (c) both make use of the Division Algorithm for polynomials and rely on degree arguments. Sometimes it is easy to forget the simple things. Note that this implies that this set of results needs the fact that A is entire.

Problem 8.4.3

Problem 3 Notes

Solution

The theory of bilinear forms is useful in many different fields covered by the written exam. Depending on the distribution of instructors for your first year courses, this type of question may appear in the context of topology or functional analysis. Part (b) is pretty straightforward: you simply wield the non-degeneracy as a sledgehammer until the details work out. Part (c) follows pretty naturally once you remember to write it as an induction proof. Again, the non-degeneracy provides the centerpiece, although the fact that you can rescale basis elements also plays its part.

Problem 8.4.4

Problem 4 Notes

Solution

Hooray for the Frobenius automorphism. This should always be your first technique to try out on finite field Galois problems. Part (b) can feel a little strange if you haven't spent much time with inseparable extensions, since it might not be clear how much leverage you obtain from assuming non-separability. However, the result follows quickly from the derivative formulation, so if you get to that step everything should work out fine.

Problem 8.4.5

Problem 5 Notes

Solution

A simpler way to attack this problem on the actual exam is to note that the desired group must be non-abelian and start trying the small ones. The permutation representation and cycle types are your friend.

Alternatively, we can really start by saying $|G| \geq 6$ (or $\frac{1}{|G|} \leq \frac{1}{6}$). Now we can get a better upper bound on the order of H. Since $|H| \leq |K|$, we have

$$1 = \frac{1}{|G|} + \frac{1}{|H|} + \frac{1}{|K|} \leq \frac{1}{6} + \frac{2}{|H|}$$

and so $|H| \leq \frac{12}{5} = 2.4$. That is, $|H| \leq 2$. As before, we know it is not possible for $|H| = 1$, so we have concluded that $|H| = 2$. Furthermore

$$1 = \frac{1}{|G|} + \frac{1}{|H|} + \frac{1}{|K|} \leq \frac{1}{6} + \frac{1}{2} + \frac{1}{|K|},$$

so $|K| \leq 3$. Now we need only check the cases when $|K| = 3$ and $|K| = 2$, as in the given solution.

For yet another solution, we can use some basic calculus. We can argue that the center of G is trivial rather easily. Once we've reduced to the case where the center of G is trivial (i.e., we have 2 non-trivial conjugacy classes), then the Class Equation forces $|G| = 1 + x_1 + x_2$. Rearranging terms, either x_1 or x_2 must be $\geq \frac{|G|}{2}$ (any non-abelian group has size ≥ 6). WLOG, say $x_1 \geq \frac{|G|}{2}$. Moreover, $x_1 \mid |G|$ by Lagrange's Theorem, and so $\boxed{x_1 = \frac{|G|}{2}}$.

Revisiting the Class Equation, $x_2 = \frac{|G|}{2} - 1$. Again $x_2 \mid |G|$ by Lagrange's Theorem. Thus $|G| = n \cdot (\frac{|G|}{2} - 1)$. Solving for $|G|$, we obtain $|G| = \frac{2n}{n-2}$. As a function of n, $|G|$ decreases for $n \geq 3$.[4] Hence $|G| = 6$ and so $G = S_3$.

Problem 8.4.6

Problem 6 Notes

Solution

This one seems a little trickier than the previous Galois problems but it really just follows the same set of steps and methods. Don't be scared of the big numbers.

[4] Just take a derivative!

Algebra Commentary

Summer 2014

Problem 8.5.1

Problem 1 Notes

Solution

Here we see the classic commutative algebra problem using the standard approach for projective modules: prove it for a free module and extend. This is one of the equivalent definition of projectivity and a nice application of some basic universal property nonsense.

It seems that most of the people who took this exam used the approach presented here (proving the lemma separately from the main result). However, depending on the level of detail in your version of Math 101, you might simply assume the result for free modules.

Problem 8.5.2

Problem 2 Notes

Solution

Probably the easiest algebra question ever asked on a written qual. As a result, we might have been so nervous about the level of this question that we supplied three different proofs on the actual exam.

Problem 8.5.3

Problem 3 Notes

Solution

This argument is equivalent to a special case of the result presented in 4.2.4. It might be easier to prove the general result and then specialize instead of working directly with the specific counting problem.

In the midst of the proof, we assumed that $QR \leq G$ (and $P(QR) \leq G$). This is a standard fact and a proof is as follows:

Proof. Since $1 \in QR$, we just need to check closure and inverses.

- (Closure) Let $a_1b_1, a_2b_2 \in QR$, where $a_1, a_2 \in Q$ and $b_1, b_2 \in R$. Since $Q \trianglelefteq G$, we have $b_1a_2b_1^{-1} = c \in Q$. Thus
$$a_1b_1a_2b_2 = a_1b_1a_2b_1^{-1}b_1b_2 = a_1cb_1b_2 \in QR$$
since $a_1, c \in Q$ and $b_1, b_2 \in R$.

- (Inverses) Let $ab \in QR$, with $a \in Q$, $b \in R$. Then $(ab)^{-1} = b^{-1}a^{-1}$. Now, by normality, $c := b^{-1}a^{-1}b \in Q$. So $(ab)^{-1} = cb^{-1} \in QR$ as $c \in Q$ and $b^{-1} \in R$.

Thus $QR \leq G$. □

Problem 8.5.4

Problem 4 Notes

Solution

This seems at first glance like a long messy problem but it really just checks the same basic Galois knowledge that the "compute the splitting field of $x^n - a$" type problems do. This problem does rely on some fairly course specific details about cyclotomic extensions and is much simpler if you know the matrix embedding for $\mathbb{Q}(\sqrt[n]{a}, \zeta_n)$.

For instance, when determining (d, e) in part (a), there was a homework problem showing that $\mathbb{Q}(\sqrt{5}) \subset \mathbb{Q}(\zeta_5)$. This is a good exercise to work through.

Problem 8.5.5

Problem 5 Notes

Solution

Part (a) is just a straightforward distinguished classes proof. In fact, it should be reasonable to argue that these extensions follow from knowledge about distinguished classes.

Part (b) is not so bad if you remember Dirichlet's Theorem and quite difficult if you don't.[5]

Problem 8.5.6

Problem 6 Notes

Solution

Noetherian ring properties and the equivalence of their definitions is definitely something you should be prepared to answer. Hilbert's basis theorem has become slightly less popular as a qual question but it is still a good proof to know.

The construction in part (a) could be formalized with an inductive proof but this is probably not necessary.

For part (b), the second proof probably contains the most intuition but the third proof is the "slickest." However, part of the exam is demonstrating what you know and one of these is a little deficient in that regard. Also, note that we did not use the integral domain structure of A here (it isn't necessary).

[5]Not to mention quite frustrating.

Algebra Commentary

Fall 2014

Problem 8.6.1

Problem 1 Notes

Solution

Just a check of the basic definitions and computations in the module view of linear algebra. Nothing scary here.

Problem 8.6.2

Problem 2 Notes

Solution

A brief trip through tensor product and universal property land. The main tool in both parts is the fact that tensor product distributes over direct sum. In the simplest case, this is stated as $A \otimes_R (B \oplus C) \cong (A \otimes_R B) \oplus (A \otimes_R C)$.

Another way of viewing part (a) is to consider a basis $(m_i)_{i \in I}$ for M and a basis $(n_j)_{j \in J}$ for N. In this view, basis element m_i corresponds to copy i of R in the direct sum expression (and similarly for n_j). Then the proof above shows that $(m_i \otimes n_j)_{(i,j) \in I \times J}$ is a basis for $M \otimes_R N$.

Problem 8.6.3

Problem 3 Notes

Solution

This is a group theory result right out of Dummit and Foote. Fairly standard counting argument and lots of different ways to repackage the proof.

Problem 8.6.4

Problem 4 Notes

Solution

Another straightforward Galois problem with an extra interesting computation in part (d).

Problem 8.6.5

Problem 5 Notes

Solution

Part (a) is just a simple check of field extension basics. Part (b) requires you to know something about distinguished classes and perform some middle–school level algebra, not necessarily an easy task on the exam itself. Be sure to note the actual structure of the proof here, showing that the individual summands of α lie in $\mathbb{Q}(\alpha)$ is a great way to tackle these problems that doesn't require you to compute the minimal polynomial. [6]

[6] Which since you asked is: $f(x) = x^6 - 6x^4 - 10x^3 + 12x^2 - 60x + 17$.

Problem 8.6.6

Problem 6 Notes

Solution

Part (a) requires you to think a little bit about what separability means and what can go wrong in characteristic p fields. The Frobenius map is always your friend.

Part (b) is very much a Gauss' Lemma problem. Just follow your nose (and the hypotheses of the lemma).

Algebra Commentary

Summer 2015

Problem 8.7.1

Problem 1 Notes

Solution

This is a nice check of basic universal mapping properties and quotient modules. Pretty straightforward as long as you don't get lost in the notation.

Since the S^{-1} functor and the $S^{-1}R \otimes_R \cdot$ functor are naturally equivalent, part (b) is equivalent to showing that $S^{-1}R$ is flat.

Problem 8.7.2

Problem 2 Notes

Solution

Part (a) is again mostly just a definition check. Part (b) is a little sneaky if you don't see the trick right away. Slow down and think about the definitions. Reasoning from part (a) and thinking about the natural free module structure should point you in the right direction.

Problem 8.7.3

Problem 3 Notes

Solution

This problem has become a favorite recently, on both oral and written exams. Although it seems like there are a lot of moving parts here, the key ideas are really just linear algebra and the Sylow theorems.

Problem 8.7.4

Problem 4 Notes

Solution

One of the shortest proofs to ever be on the written exam. The intuition plays on the relationship between finite and algebraic extensions of fields and the algebraic closure of $\mathbb{C}$.

It is very easy to fall into a trap with this problem. Below is one incomplete solution to this problem:

Proof. Let $\alpha \in R$ and let $n = \dim_{\mathbb{C}} R$. Then the elements $1, \alpha, \ldots, \alpha^n \in R$ are a dependent set, so there are constants $c_i \in \mathbb{C}$, not all zero, such that

$$c_0 + c_1\alpha + \cdots + c_n\alpha^n = 0.$$

Thus α is the root of the non-zero polynomial $c_0 + c_1 x + \cdots + c_n x^n$, so $\mathbb{C}(\alpha)/\mathbb{C}$ is an algebraic extension of fields. But $\mathbb{C}$ is algebraically closed, so $\mathbb{C}(\alpha) = \mathbb{C}$. Thus $\alpha \in \mathbb{C}$ and so $R = \mathbb{C}$. □

Now, run the same argument with $\mathbb{C} \times \mathbb{C}$. Does it fail? Why?

Problem 8.7.5

Problem 5 Notes

Solution

The most interesting direction of this problem is (iii)$\to$(i), which hinges on the containment relations of finite fields with the same characteristic.

Problem 8.7.6

Problem 6 Notes

Solution

This problem requires some fairly messy computations as far as Galois problems go. However, discovering the roots only requires the quadratic formula and from there determining the Galois group structure is routine if not pleasant.

Algebra Commentary

Fall 2015

Problem 8.8.1

Problem 1 Notes

Solution

The first two parts are just basic module facts and definitions. The third part requires understanding the consequences of part (b) and using the elementary *integer* column operations to justify the possible submodules. The real key is remembering that column operations change the *basis* for the image but not the image itself.

Problem 8.8.2

Problem 2 Notes

Solution

This problem might look a little mysterious at first but once you identify that it is really a question about canonical forms everything follows nicely since forms are only defined up to similarity class.

Problem 8.8.3

Problem 3 Notes

Solution

For part (a) the assumption of simplicity is our big tool to use. The map should also look familiar from Galois theory. Part (b) is an easy application of (a).

Problem 8.8.4

Problem 4 Notes

Solution

This is a classic oral qual question from the old system. Properties of the commutator aren't always covered in the algebra courses but if this question appears on your qual you should have seen the material in class. Part (b) is in some sense just a check that you know the fundamental theorem of Galois theory given part (a).

Problem 8.8.5

Problem 5 Notes

Solution

Computational Galois theory without having to count the degrees of the extensions. Luckily, it is easy to practice lots of problems like this if you are rusty.

Also, there is a more general result lurking in the background here. Assume K is the splitting field of an irreducible separable polynomial f over F. If f has a real and a complex root, then $\mathrm{Gal}(K/F)$ is nonabelian.

Problem 8.8.6

Problem 6 Notes

Solution

The setup for this problem is a little complex but once you have all the pieces straight in your head, it's not so bad.

An alternate solution, for those more comfortable with Galois theory, is the following:

Proof. Since K/F is Galois, every automorphism of L over F can be obtained by extending an automorphism of K.

- To lift the identity, $\mathrm{id} : K \to K$, to $L = K(\sqrt{b})$, we consider roots of $x^2 - b$ in L. Since $x^2 - b$ has two roots in L, there are two lifts of id to automorphisms of L.
- To lift $\sigma : K \to K$, we consider the roots of $\sigma(x^2 - b) = x^2 - \sigma(b)$ in L. Then L/F is Galois if and only if $\sqrt{\sigma(b)} \in L$.

Since the quadratic extensions of K are determined up to $K^\times/K^{\times 2}$,

$$\sqrt{\sigma(b)} \in L \Leftrightarrow \quad K(\sqrt{b}) = K(\sqrt{\sigma(b)}) \Leftrightarrow b\sigma(b) \in K^{\times 2}.$$

$\square$

Algebra Commentary

Summer 2016

Problem 8.9.1

Problem 1 Notes

Solution

See notes here: 8.4.1.

Problem 8.9.2

Problem 2 Notes

Solution

Part (a) is a counting argument, relying on a consequence of the Class Equation. For part (b) we need the smallest non-abelian p-group and luckily 8 works.

Problem 8.9.3

Problem 3 Notes

Solution

This proof perfectly carries over to the more general setting of finitely generated modules over a PID, with "infinite order" replaced with "torsion-free".

The proof looks quite technical, but the general idea is much simpler than it looks. First, we take advantage of the structure theorem for finitely generated abelian groups (or modules over a PID) to write x and y in terms of some free coordinates and torsion coordinates. Then, we throw away all but one coordinate such that the coordinate we keep is (i) a free coordinate and (ii) non-zero. We conclude by using the fact that $\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}$ is free of rank one (more succinctly, $R \otimes_R R \cong R$) to show that the non-zero free coordinates of x and y remain non-zero when we take the tensor product.

Ultimately, this is a fun problem that just boils down to showing you can avoid the torsion subgroup. The approach looks a little different than most proofs that elements in a tensor product are non-zero, but overall you are still trying to find a basis element to which to map.

Problem 8.9.4

Problem 4 Notes

Solution

The key idea for part (a) is to be able to move comfortably between different choices of how to view the coefficients of the polynomial ring. Adjoining the variables one at a time is often a great technique.

Part (b) just requires the Hilbert Basis Theorem and some familiarity with types of ideals. It's useful to remember that a domain is a UFD if and only if (i) every element has (not necessarily unique) factorization into irreducibles and (ii) every irreducible is prime. Furthermore, to prove that a quotient ring has some property, you can prove that the ideal has some corresponding property. Another example of this is that R/I is a field if and only if I is maximal (assuming R is commutative).

Part (c) boils down to using Gauss's Lemma and remembering when a polynomial ring is a PID.

Problem 8.9.5

Problem 5 Notes

Solution

See notes here: 8.3.4.

Problem 8.9.6

Problem 6 Notes

Solution

This problem is just a chance to show off how much you know about Galois theory. The difference between an acceptable answer for the qual and the "correct" answer is particularly large for this problem.

Algebra Commentary

Fall 2016

Problem 8.10.1

Problem 1 Notes

Solution

A basis free proof of oart (a) is also not so hard to write down. In order to come up with an example for part (b), your first thought should be to find the simplest possible projections and subspaces. Part (a) reminds you that projections don't commute generically.

Problem 8.10.2

Problem 2 Notes

Solution

These types of problems are becoming more popular and you should be familiar with the computations and ideas required.

Also, we can count $\mathrm{GL}_n(\mathbb{F}_q)$ in much the same way: there are $q^n - 1$ options for the first column, $q^n - q$ for the second column (because it can't be a multiple of the first column, and there are q of those), $q^n - q^2$ for the third column (because it can't be in the span of the first two columns, which comprises q^2 vectors, and so on: there are $q^n - q^{i-1}$ options for column i, because we must exclude the q^{i-1} vectors in the span of the first $i-1$ columns. Therefore $|\mathrm{GL}_n(\mathbb{F}_q)| = (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$. Using this, we can count $\mathrm{SL}_n(\mathbb{F}_q)$ too: $\mathrm{GL}_n(\mathbb{F}_q)/\mathrm{SL}_n(\mathbb{F}_q) \cong \mathbb{F}_q^\times$, so $|\mathrm{SL}_n(\mathbb{F}_q)| = |\mathrm{GL}_n(\mathbb{F}_q)|/(q+1)$.

Problem 8.10.3

Problem 3 Notes

Solution

This is all about tying definitions together and the primary decomposition for modules over a PID. Usually we prove the direct sum conditions but here we are using it.

Problem 8.10.4

Problem 4 Notes

Solution

Classic ring theory.[7] Although we were glib about what elements of $\mathbb{Z}[\sqrt{-5}]$ were units and irreducibles, it doesn't seem as though we *had* to prove those parts of the problem.

Problem 8.10.5

Problem 5 Notes

Solution

[7] With a slight hint of number theory.

Back to slightly more traditional Galois problems. This has the slight twist that you need to use the FTGT to identify the groups. Don't think too hard. In part (c), note that we found H by taking "what was left" to get L after finding the subextension which gave us S_3 as a Galois group.

Problem 8.10.6

Problem 6 Notes

Solution

This is a classic problem and one of our personal favorites. At its heart this problem is really just a collection of counting problems, making use of your knowledge of extension theory to get the (simple) bounds necessary for the contradictions.

Also, the requirement in part (c) that $n \geq 4$ is necessary. Indeed, $3 \cdot \varphi(3) = 6$ and the Galois group of $x^3 - 2$ (over $\mathbb{Q}$) is S_3.

Algebra Commentary

Summer 2017

Problem 8.11.1

Problem 1 Notes

Solution

For part (a), everyone immediately guessed the quaternions: 8 element group, John Voight wrote the question, etc. oops!

For part (b), it's tempting to build a section to the short exact sequence and move on. However, it's essential to remember that this SES takes place in the category of groups (and not abelian groups or modules). In this setting, a SES splits if and only if there is a retraction.

Problem 8.11.2

Problem 2 Notes

Solution

You could element chase part (a) if you really wanted to, but it probably isn't worth the pen ink.[8]

There are two other possible interpretations of part (b), depending on how charitable you are feeling to the examiner. Excluding the implicit assumption noted in the footnote, you might choose to take your zero blocks to have size zero[9] in which case the result is vacuously true in any basis. Alternatively, you may instead assume that the blocks must all have non-zero dimensions, in which case the problem is false as stated.

Problem 8.11.3

Problem 3 Notes

Solution

As long as you remember the definition of the dual space, you should be good to go.

Problem 8.11.4

Problem 4 Notes

Solution

An alternate method to show that x^4+1 is irreducible over $\mathbb{Z}[x]$ is by making use of the automorphism of $\mathbb{Q}[x]$ determined by $f(x) \mapsto f(x+1)$. In our particular case, $(x+1)^4+1$ is irreducible by Eisenstein's Criterion ($p=2$), and hence the original polynomial must also be irreducible.

Problem 8.11.5

Problem 5 Notes

Solution

[8]You are writing in pen, right?

[9]Not explicitly disallowed.

The first two parts should be familiar to anyone who has made it this far in the book. Part (c) is a nice twist on the standard material and works a fun Sylow argument into the exam.

Problem 8.11.6

Problem 6 Notes

Solution

Depending on the exact topics covered in your class, the result of parts (a) and (c) may have been discussed extensively. Either way, the real subfield and cyclotomic field properties are useful. You can discover part (b) by thinking about $\mathbb{Q}(i)$, if it is not clear at first how to proceed. Finally, in a very exciting fashion, constructibility finally makes an appearance on the exam!

9

Analysis Commentary

Analysis Commentary

Summer 2012

Problem 9.1.1

Problem 1 Notes

Solution

It's important to know the equivalent definitions of continuity by heart. This problem doesn't require any specialized knowledge and it is considered basic real analysis (thus not generally covered in the first-year courses). Indeed, the proof is a matter of following the definitions through.

Problem 9.1.2

Problem 2 Notes

Solution

Both parts of this problem are fairly standard results in analysis. For part (b), there are a couple steps that require additional justification.

Firstly, we blithely write that a compact subset K of Ω is contained in a compact set K', realized as a union of disks *of the same radius*. For a sketch of this fact, write $f(z) = d(z, K)$. So f is a continuous function on Ω and thus, by compactness, $f(z) = d(z, K)$ is realized by some $w \in \Omega$. Define $K' = \bigcup_{z \in K} \overline{D(z, r)} = f^{-1}([0, r])$. Then K' is closed and bounded, hence compact.

Secondly, for the Cauchy estimates, it's better to write out the details here. In particular, remember that the derivative is linear and so

$$|f_n' - f'| = |(f_n - f)'| < (\varepsilon\rho) \cdot 1 \cdot \rho^{-1} = \varepsilon$$

where M is taken to be $\varepsilon\rho$.

Problem 9.1.3

Problem 3 Notes

Solution

Both parts of this problem require mucking around with εs and proving the desired results. For part (b), the strategy is a standard one: first find a candidate for the limit of a Cauchy sequence (using completeness of $\mathbb{R}$ or $\mathbb{C}$), usually some kind of pointwise convergence, and then prove the additional properties separately.

Problem 9.1.4

Problem 4 Notes

Solution

The Riesz–Fréchet and Closed Graph Theorems are both extremely important to keep in your back pocket. Both are typically taught in the functional analysis course and this is a standard homework problem in that course.

The fact that we should use the Closed Graph Theorem in part (b) is certainly not immediate – there's nothing obvious from the statement of the problem that it should be involved at all. However, since we're showing the boundedness of T, there are only so many choices available.

Problem 9.1.5

Problem 5 Notes

Solution

The introduction to Ullrich's *Complex Made Simple* covers almost all of this problem. For the sufficient condition, note that the assumptions are rather extreme but they get the job done. Also, it's perhaps better to internalize the idea that the Cauchy–Riemann equations are the necessary structure to the make the real derivative of $f_{\mathbb{R}}$ into a $\mathbb{C}$-linear function.

Problem 9.1.6

Problem 6 Notes

Solution

There are no substantive comments to be made about this problem. Go forth and prove things!

Analysis Commentary

Fall 2012

Problem 9.2.1

Problem 1 Notes

Solution

The proof that points can be separated by linear functionals is a standard corollary to the Hahn–Banach Theorem. There isn't a lot going on in this problem.

Problem 9.2.2

Problem 2 Notes

Solution

The problem itself is fairly standard for the Residue Theorem applications. However, it is possible to go through a lot of excessively long computations in this problem. For the written exam, that's exactly something to avoid when possible. For instance, the full proof that $\int_{\alpha_R} f(z)\, dz \to 0$ as $R \to \infty$ requires more actual computing but that takes away from the essential solution.

Problem 9.2.3

Problem 3 Notes

Solution

This is a standard problem in real analysis and the details can trip you up. Try writing down the general idea and focus on constructing the appropriate examples (of which there should be many).

Problem 9.2.4

Problem 4 Notes

Solution

While we didn't exactly use the theorem to determine the radius of convergence, it's unclear what they were expecting otherwise.

Problem 9.2.5

Problem 5 Notes

Solution

It's possible that there is a cleaner way to solve this problem than the solution presented here. However, if we ignore the "back and forth" nature of this solution, this problem comes down to using general properties of measures. On the exam, it comes down to already knowing what you have to do.

Problem 9.2.6

Problem 6 Notes

Solution

This is another standard real analysis problem. Not a whole lot to it.

Analysis Commentary

Summer 2013

Problem 9.3.1

Problem 1 Notes

Solution

This problem is mostly about understanding the Cauchy–Riemann equations. It can help to relabel the real and imaginary parts of g and h to help track whether the equations hold, but that's a personal preference.

Problem 9.3.2

Problem 2 Notes

Solution

There is nothing intentionally tricky about this problem, so long as you remember the limit point definition of closed sets.

Problem 9.3.3

Problem 3 Notes

Solution

This is a straightforward proof using one of the major theorems in measure theory.

Problem 9.3.4

Problem 4 Notes

Solution

This is another straightforward problem. It doesn't rely on any heavy machinery or difficult techniques. Know your definitions and you will get through this one.

Problem 9.3.5

Problem 5 Notes

Solution

Unlike the remainder of the problems on this analysis exam, this problem is all about the tricky details. Knowing how to create your subsequences for case (III) is intuitively clear and horrible to write down correctly.

Problem 9.3.6

Problem 6 Notes

Solution

For part (a), the Cauchy sequence argument is rather standard. The rest of the problem relies on basic facts about bounded linear maps.

Analysis Commentary

Fall 2013

Problem 9.4.1

Problem 1 Notes

Solution

Since we have an entire function and want to show that it's constant, the most natural result to use is Liouville's Theorem. That is, we need to show the map is bounded. This is always possible on compact sets but we have to use the additional assumption to get a bound elsewhere.

Problem 9.4.2

Problem 2 Notes

Solution

The essential intuition for this problem is that the condition on norms holds if and only if

$$(x, y) + (y, x) = 0.$$

Problem 9.4.3

Problem 3 Notes

Solution

This problem is all about knowing what can go wrong with integration. Working over a simply-connected domain is usually a good sign; holes are a bad sign.

Problem 9.4.4

Problem 4 Notes

Solution

Showing that V is a bounded operator is a standard argument. To show that it has no eigenvalues, we need to pull out the Fundamental Theorem of Calculus. It helps that this problem came from a homework problem in Math 113.

Problem 9.4.5

Problem 5 Notes

Solution

Here's your chance to use the Dominated Convergence and Monotone Convergence Theorems! Once you've made this realization, the proofs follow from the normal tricks.[1]

[1] Unless you deduce the wrong inequalities, at which point you're hopelessly stuck in a weird loop of doom.

Problem 9.4.6

Problem 6 Notes

Solution

This problem also showed up as a homework problem in Math 113. The general theme is to use u-substitution to figure out the computations. However, both the Cauchy–Schwarz(–Bunyakovsky) inequality and Fubini's Theorem make an appearance.

Analysis Commentary

Summer 2014

Problem 9.5.1

Problem 1 Notes

Solution

Parts (a) and (b) are Proposition 2.7 in Folland's textbook on real analysis. It is good to remember that you can prove a function f is measurable by, for instance, showing that $f^{-1}((a,\infty])$ is measurable for every $a \in \mathbb{R}$ — or, more generally, by showing that $f^{-1}(B)$ is measurable for every B in a collection of sets that generates the σ-algebra of measurable sets in the codomain of f.

For part (b), note that we assumed the proof of the analogous result for infimums. This is perfectly acceptable and certainly expected in this case.

For part (c), the requirement of "starting with the definition of measurable function" is the reason we did not use the following simpler argument: $f - g$ is measurable, and $E = (f-g)^{-1}((0,\infty])$.

Problem 9.5.2

Problem 2 Notes

Solution

We prove the statement first for characteristic functions, then for simple functions, then for arbitrary functions. This is an extremely useful technique in measure theory, especially the part where we write the arbitrary function as an increasing limit of simple functions and use the Monotone Convergence Theorem.

Problem 9.5.3

Problem 3 Notes

Solution

Always remember Liouville's Theorem.

Be mindful of the difference between the statements "$\lim_{z\to\infty} f(z) = \infty$" and "$\lim_{z\to\infty} f(z)$ does not exist". In the former, $f(z)$ *stays* arbitrarily large as z becomes arbitrarily large; in the latter, $f(z)$ is unbounded but keeps getting small again as z becomes arbitrarily large. If the limit did not exist, then $f(1/z)$ would not have a pole at 0 like we needed it to, but rather an essential singularity. From the perspective of the Riemann sphere $\overline{\mathbb{C}}$, the definition of converging to ∞ is the same as the definition of converging to any finite point.

Problem 9.5.4

Problem 4 Notes

Solution

This is a standard Banach space argument: find the candidate limit and use assumptions to force it to work out.

Problem 9.5.5

Problem 5 Notes

Solution

To prove that M is well-defined, in addition to proving that $M(f)$ really falls in the codomain, we also had to prove that $M(f)$ does not depend on the choice of representative for f. Recall that technically L^p spaces consist of equivalence classes of functions, where two functions are equivalent if they are equal almost everywhere in the domain. This technicality was harmless in this problem, but it is worth remembering.

We proved that $\|M\| \leq 1$; in fact, $\|M\| = 1$. More generally, given h continuous on $[0, 1]$, we can define $M(f) = hf$, and then M is a bounded linear operator on $L^2(0, 1)$ with $\|M\| = \|h\|_\infty$.

Problem 9.5.6

Problem 6 Notes

Solution

Notice that δ is the distance from x to M. In part (a), we show that, when M is closed and $x \notin M$, this distance is strictly positive.

Part (b) is much easier in a Hilbert space. Indeed, letting $x_\perp$ be the orthogonal projection of x onto $M^\perp$, we take our linear functional to be $z \mapsto \frac{1}{\delta}(z, x_\perp)$. In this problem, we use the Hahn–Banach theorem to prove it in much more generality — it is not even required for V to be complete.

Analysis Commentary

Fall 2014

Problem 9.6.1

Problem 1 Notes

Solution

Parts (a) and (b) are fairly straightforward applications of the convergence theorems from measure theory. Part (c) requires understanding why we might expect the equation to fail and using that to build a reasonable example.

Problem 9.6.2

Problem 2 Notes

Solution

This is a standard homework problem. The "trick" in this problem is realizing that you can partition the domain into the sets S_1 and S_2. At first glance, it's not immediately obvious that this is what you should be doing.

Problem 9.6.3

Problem 3 Notes

Solution

The result for part (a) is a special case of Jordan's Lemma, so if you are familiar with this result then the proof takes only one line. If you are not familiar with this result the proof of the general case follows along exactly the same path as the problem presented here, replacing $\frac{1}{z^4}$ with any other function, possibly with poles, that goes to zero as $z \to \infty$. Part (b) then just requires you to join up the two semicircles and apply the Residue Theorem.

Problem 9.6.4

Problem 4 Notes

Solution

This is a straightforward problem. No special commentary required.

Problem 9.6.5

Problem 5 Notes

Solution

The main issue with this problem is the potential of getting lost in the Sea of Horrible Subscripts. If you can navigate that morass, then you'll be fine.

There's another approach to this problem which is nearly identical. Start by noting that $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space. Then it suffices to show that c is a closed subspace of ℓ^∞. So we take a convergent sequence in c and show that the limit is also a convergent sequence in $\mathbb{C}$. Try it and see what appeals to you more.

Problem 9.6.6

Problem 6 Notes

Solution

There isn't a lot to say about this problem. You need to remember some facts about how the range of the adjoint relates to the original operator and, otherwise, follow your nose. Wherever it may lead you.

Analysis Commentary

Summer 2015

Problem 9.7.1

Problem 1 Notes

Solution

See the comments to problem 2(a) from Summer 2012 (9.1.2).

Problem 9.7.2

Problem 2 Notes

Solution

Much like other problems based on applying Liouville's Theorem, we need to show that a particular function is bounded. In this case, the assumption on f tells us that we can start looking from degree 3 of its Taylor expansion about 0.

Problem 9.7.3

Problem 3 Notes

Solution

See the comments to problem 5 from Fall 2012 (9.2.5).

Problem 9.7.4

Problem 4 Notes

Solution

The really interesting part of this problem is (c). Showing that T is bounded relies on the realization that we should employ one of the "big" results in functional analysis (which typically require at least a Banach space). In this case, we actually use two: the Uniform Boundedness Principle and the Hahn–Banach Theorem.

Problem 9.7.5

Problem 5 Notes

Solution

The solutions to each part of this problem are "standard" for the setting. They may be more familiar for the setting of bounded linear operators, but the proofs are exactly the same. Part (a) is known as the Carl Neumann Criterion, part (b) is a corollary to this criterion, and part (c) is the consequence on spectra.

Problem 9.7.6

Problem 6 Notes

Solution

This problem is nicely constructed to give you all the tools you need for part (c). The techniques used here rely on properties of Hilbert spaces and the adjoint but nothing fancier than those.

Analysis Commentary

Fall 2015

Problem 9.8.1

Problem 1 Notes

Solution

This result can be a consequence of Montel's Theorem, which typically concludes that the family of maps $\{f_n\}$ has a subsequence which converges uniformly on compact sets. Then we obtain analyticity by, for instance, problem 2a from Summer 2012 (5.1.2).

Problem 9.8.2

Problem 2 Notes

Solution

The result in part (b) is known as the Casorati–Weierstrass Theorem. In the midst of the proof, we note that g is holomorphic at z_0 without much justification. However, this is a good opportunity to read up on analytic continuation. For our particular situation, g was analytic at z_0 because g was bounded on a neighborhood of z_0.

Problem 9.8.3

Problem 3 Notes

Solution

This problem would be substantially more difficult if not for the provided hint. Showing that $\mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ is not especially interesting or difficult, but the reverse inequality takes some care.

Problem 9.8.4

Problem 4 Notes

Solution

The distinction between pointwise convergence and convergence in norm is particularly stunning in this problem.

Problem 9.8.5

Problem 5 Notes

Solution

In the solution, we used the fact that a linear functional defined on Q extends to a bounded linear functional such that $\varphi_{n_j}(x) \to \varphi(x)$ for all $x \in E$. It is unclear whether this result could reasonably be assumed or if it was supposed to be proven in the middle of the problem. A complete proof of this fact is presented below.

Theorem 9.8.1 *Let E be a Banach space and Q a dense subset. Let $\{\varphi_n\}$ be a bounded sequence in E^* (with bound M) and assume $\{\varphi_n(q)\}$ is convergent for all $q \in Q$. Then $\{\varphi_n(x)\}$ is convergent for all $x \in E$ and the limit $\varphi(x)$ defines a bounded linear functional on E.*

Proof. We will first show that $\{\varphi_n(x)\}$ is Cauchy for all $x \in E$. Let $\varepsilon > 0$ and choose, by density of Q, some $q \in Q$ such that $\|x - q\| \leq \varepsilon/4M$. Then, for all n,

$$|\varphi_n(x) - \varphi_n(q)| \leq \|\varphi_n\| \, \|x - q\| \leq M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{4}.$$

Since $\{\varphi_n(q)\}$ is convergent, $\exists N$ such that, for all $n \geq N$, we have

$$|\varphi_n(q) - \varphi(q)| \leq \frac{\varepsilon}{4}.$$

Thus, for $m, n \geq N$, we have

$$\begin{aligned}
|\varphi_n(x) - \varphi_m(x)| &= |\varphi_n(x) - \varphi(q) + \varphi(q) - \varphi_m(x)| \\
&\leq |\varphi_n(x) - \varphi(q)| + |\varphi(q) - \varphi_m(x)| \\
&\leq |\varphi_n(x) - \varphi_n(q)| + |\varphi_n(q) - \varphi(q)| + |\varphi(q) - \varphi_m(q)| + |\varphi_m(q) - \varphi_m(x)| \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Hence $\{\varphi_n(x)\}$ is Cauchy. Since E is complete, this means that $\varphi_n(x) \to \varphi(x) \in \mathbb{F}$.

Thus it remains to show that φ is a bounded linear functional. For linearity, we use the fact that φ_n is linear and the uniqueness of convergence. For boundedness, note that $\|\varphi\| \leq M$ since, for all n, $\|\varphi_n\| \leq M$. □

Problem 9.8.6

Problem 6 Notes

Solution

The shift maps are typical examples of the given behavior – know them.

Analysis Commentary

Summer 2016

Problem 9.9.1

Problem 1 Notes

Solution

This result is known as the Reverse Fatou Lemma. The solution given, using Fatou's Lemma, is standard, but it is also possible to simply use the proof of Fatou's Lemma (either direct or with the Monotone Convergence Theorem) using $g - f_n$ at every step.

It's important to note that the logical starting place for the inequality is where we applied Fatou's Lemma. The remaining equalities follow from there naturally.

Problem 9.9.2

Problem 2 Notes

Solution

The major things to remember in this problem is how the product σ-algebra is defined and that integration with respect to the counting measure (on a countable space) is summation.

Problem 9.9.3

Problem 3 Notes

Solution

This problem is all about the different theorems in complex analysis:

- Part (a) uses Cauchy's Theorem.
- Part (b) uses the Identity Theorem. This is equivalent to the well-known fact that the zeroes of a nonzero holomorphic function are isolated.
- Part (c) uses Liouville's Theorem.
- Part (d) uses the Residue Theorem.

Problem 9.9.4

Problem 4 Notes

Solution

Getting through this problem relies heavily on the fact that norm convergence implies weak convergence. The major hint that we should be using the Closed Graph Theorem to show boundedness is that we have Banach spaces and an additional hypothesis that has to deal with limits.

Problem 9.9.5

Problem 5 Notes

Solution

Part (b) of this problem is manageable without having many realizations about what to do for this problem. However, part (a) and (c) are filled with traps. In part (a), we should use the Dominated Convergence Theorem because we can get a constant bound on $|T(f)|$ (this is something of a standard procedure). Doing this carefully requires some obnoxious details, as shown in the solution.

For part (c), it's important to settle on the "correct" definition of compactness. From there, the Arzelà–Ascoli Theorem is actually a natural candidate. To apply it, uniform boundedness is easy and equicontinuity is hard. The proof that the sequence $\{T(f_n)\}$ is equicontinuous relies on K being uniformly continuous (otherwise we can't get anything in terms of $|x_1 - x_2|$) which is easy if you remember the Heine–Cantor Theorem and a roadblock otherwise.

Problem 9.9.6

Problem 6 Notes

Solution

This result may be generalized by weakening our assumptions. Indeed, the result holds if $\{f_n\}_{n\in\mathbb{N}}$ is monotonic (either non-increasing or non-decreasing) and f is a continuous function such that $f_n \to f$ pointwise. If $\{f_n\}_{n\in\mathbb{N}}$ is non-decreasing, the result is commonly referred to as Dini's Theorem. Slight modifications of the proof we present yields a proof of the distinct variants.

Analysis Commentary

Fall 2016

Problem 9.10.1

Problem 1 Notes

Solution

Part (a) is an exact duplicate of Summer 2013 (problem 3). We have reproduced the solution here because there are other parts to address in this problem.

Ultimately, this parts (a) and (b) of this problem are all about the Monotone Convergence Theorem. Part (c) requires some care, however. Especially during the stressful time that is the exam, one might not see that the containment is with respect to *different* measures. Also, g is still in $L^+(X, \mathcal{M})$ but it has the added property of being square-integrable.

Problem 9.10.2

Problem 2 Notes

Solution

This problem is all about properties of σ-algebras and how Lebesgue measurable sets are defined. Ultimately, there are no fancy arguments.

Problem 9.10.3

Problem 3 Notes

Solution

For part (b), the only additional hypothesis is that you have a maximum on the circle. Given that f is analytic (hence continuous) on the entire disk and circles are compact, this isn't a very meaningful assumption – it's always true. Thus we only truly require the hypotheses for the Cauchy Integral Formula.

Problem 9.10.4

Problem 4 Notes

Solution

This is a standard proof in functional analysis.

Problem 9.10.5

Problem 5 Notes

Solution

Establishing linearity of T first gives us additional leverage with other results (here, with the Closed Graph Theorem). Once T is a bounded linear map, uniqueness of the adjoint establishes the same properties for S.

Problem 9.10.6

Problem 6 Notes

Solution

The main observation here is that we want to apply Arzelà–Ascoli. To see this, we rely on the definition of sequential compactness. For metric spaces, this is equivalent to the topological notion of compactness (why?) but sequences tend to be the better notion with function spaces.

Analysis Commentary

Summer 2017

Problem 9.11.1

Problem 1 Notes

Solution

With the exception of the phrasing of the question, this is a fairly standard type of complex analysis problem.

Problem 9.11.2

Problem 2 Notes

Solution

The only potential problem with the argument for part (b) is when u is not ν-integrable, meaning that $\int_Y u^+ \, d\nu = \int_Y u^- \, d\nu = +\infty$. But this happens if and only if $\displaystyle\int_X u^+ \circ f \, d\mu = \int_X u^- \circ f \, d\mu = +\infty$, so we conclude that u is ν-integrable if and only if $u \circ f$ is μ-integrable.

Problem 9.11.3

Problem 3 Notes

Solution

There are lots of possible examples for part (a), building on a similar theme. Depending on your instructor, they may not appreciate the "cleverness" so frequently displayed by slightly frazzled graduate students[2] and writing out the standard (if boring) example is likely to be better for your score. Parts (b) and (c) require some ε chasing but nothing too strenuous.

Problem 9.11.4

Problem 4 Notes

Solution This is a nice sequence of problems that build on each other. Anytime you see the $*$ appearing, you should start to think about how to work out the problem in terms of inner products (which have all sorts of nice properties).

Problem 9.11.5

Problem 5 Notes

Solution

Note how each of the assumptions is used throughout the proof. This type of contradiction argument by Hahn–Banach is perhaps not the most common use, but it is a nice one to have in the back of your mind. Especially since existential statements can make room for counterexamples like this one.

[2] As exemplified by the footnotes prepared by the authors of this text.

Problem 9.11.6

Problem 6 Notes

Solution

This exam concludes with a nice straightforward definition check and application of Arzelà–Ascoli. Along the way, you get to use the FToC!

10

Topology Commentary

Topology Commentary

Summer 2012

Problem 10.1.1

Problem 1 Notes

Solution

This is equivalent to Theorem 1.38 in Hatcher's Algebraic Topology. The proof falls out of knowing the definitions.

Problem 10.1.2

Problem 2 Notes

Solution

The tools that you might have available to address this problem will depend heavily on the instructor of your course. While (2) specifically relies on the Hairy Ball Theorem, there are other means of answering (3) or (4) (for instance, using their Lie group structures or explicitly constructing the nowhere vanishing vector field).

Problem 10.1.3

Problem 3 Notes

Solution

See Hatcher page 134 for a description of the properties of the degree map on homology, including a proof that the degree of a reflection is -1. (The general idea is to start on S^0 and build up to S^n by suspending; an operation that preserves degree.)

Problem 10.1.4

Problem 4 Notes

Solution

Some professors will expect a derivation of the homology of a torus using CW complexes or the Eilenberg–Steenrod axioms instead of Mayer–Vietoris. For this problem, it's important to remember the Mayer–Vietoris sequence *and* what the maps in that sequence represent. While many examples do not require the explicit maps, this is one that doesn't work without knowing the loops.

Problem 10.1.5

Problem 5 Notes

Solution

Perhaps the easiest topology question ever asked on a written exam.

Problem 10.1.6

Problem 6 Notes

Solution

This problem is an exercise in the properties of differential forms. It also contains a seemingly unintuitive step of adding 0 but this is a matter of knowing that the goal is to show that $u \wedge v - u' \wedge v'$ is a coboundary. Basically, the extra terms are thrown in to make this apparent.

Topology Commentary

Fall 2012

Problem 10.2.1

Problem 1 Notes

Solution

Don't be scared of the coordinates! Sometimes you just have to jump into the messy details.

Problem 10.2.2

Problem 2 Notes

Solution

This problem boils down to remembering definitions, computing a pushforward, and recognizing the result of the computation. Unfortunately, recognizing each of those steps is much more easily done in hindsight.

Problem 10.2.3

Problem 3 Notes

Solution

The essential observation is noted in the solution. That is, as with the Stokes' Theorem in calculus, it can be extremely useful to replace a complicated surface with a simple one which has the same boundary.

Problem 10.2.4

Problem 4 Notes

Solution

See the comments for problem 4 in the Summer 2014 exam (10.5.4).

Problem 10.2.5

Problem 5 Notes

Solution

The basic strategy, since Z is connected, is to show that the agreement set is clopen. In doing so, it's possible to get a little lost in the details (draw a picture to help keep track!).

Problem 10.2.6

Problem 6 Notes

Solution

Topology Commentary

Summer 2013

Problem 10.3.1

Problem 1 Notes

Solution

It may seem tempting to work in coordinates and show that coordinate functions are smooth. Don't do that to yourself; make this as painless as possible.

Problem 10.3.2

Problem 2 Notes

Solution

Local homology may seem like a "brand new thing" to cause extra stress. However, by definition, it's just the homology of the pair $(U, U \setminus \{p\})$. Computing this for $\mathbb{R}^n$ at 0 isn't especially hard because we know the homologies of spheres. You do remember the homology of spheres, yeah?

Problem 10.3.3

Problem 3 Notes

Solution

No commentary required.

Problem 10.3.4

Problem 4 Notes

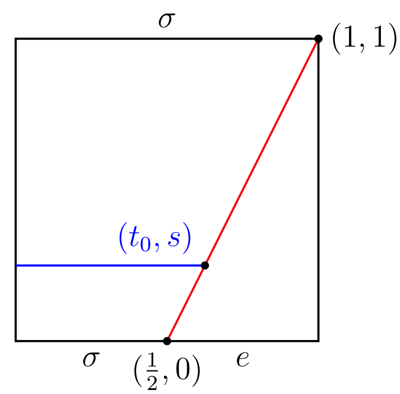
Solution

This problem boils down to knowing that you want to show $[\sigma][\tau] = [\tau][\sigma]$ and translating it to a problem on the level of G. The formulas that appear somewhat more naturally if you try drawing the homotopy square. There are classier solutions (even one line proofs), but these methods are beyond the scope of this qual.

Below we formally define f_s:

Let the coordinates of $[0,1] \times [0,1]$ be given by (t, s). Our goal is to start with $\sigma * e$ on the bottom of the square and move toward σ. A straight line from $(\frac{1}{2}, 0)$ to $(1,1)$ has the form $s = 2t - 1$ (for $t \geq \frac{1}{2}$).

Now fix s. Then we obtain the following square:



In the blue line, we need our map to cover the entire path σ. Thus, as inputs range from 0 to t_0, we want to cover $[0,1]$. Since (t_0, s) lies on the line $s = 2t-1$, we solve and obtain $\boxed{t_0 = \dfrac{s+1}{2}}$. So, for $0 \leq t \leq \frac{s+1}{2}$, let $f_s(t) = \sigma\left(\frac{2}{s+1} \cdot t\right) = \sigma\left(\frac{2t}{s+1}\right)$. For larger t, $f_s(t) = e$ works.

Below is an alternative solution that may (or may not) be appealing.

Proof. Let σ and τ be loops based at e. To show $[\sigma][\tau] = [\tau][\sigma]$, it suffices to prove $[\sigma][\tau][\sigma]^{-1} = [\tau]$.

First, we must show that $[\sigma\tau] = [\sigma][\tau]$ for any pair of loops σ, τ based at e. Phrased differently, we need to show that $\sigma\tau$ and $\sigma * \tau$ are path homotopic. This follows by considering the homotopy $F(s,t) = \begin{cases} \sigma(2s-ts)\tau(ts) & s \leq \frac{1}{2} \\ \sigma(1-t(1-s))\tau(2s-1+t(1-s)) & s \geq \frac{1}{2} \end{cases}$.

The pointwise inverse of σ, denoted σ^{-1}, is a path in G since inversion is continuous. By definition, $\sigma(s)\sigma^{-1}(s) = e$ for all s. Thus σ^{-1} is a loop based at e and $[\sigma^{-1}] = [\sigma]^{-1}$.

Consider the map defined by $H(s,t) = \sigma(st)\tau(s)\sigma^{-1}(st)$. We claim that H is a homotopy between τ and $\sigma\tau\sigma^{-1}$. Since G is a topological group, multiplication is continuous and so H is continuous.[1]

Notice that $H(s,0) = e \cdot \tau(s) \cdot e = \tau(s)$ and $H(s,1) = \sigma(s)\tau(s)\sigma^{-1}(s)$. Moreover, H is a path homotopy because $H(0,t) = e \cdot e \cdot e = e$ and $H(1,t) = \sigma(t) \cdot e \cdot \sigma^{-1}(t) = e$. Hence

$$[\tau] = [\sigma\tau\sigma^{-1}] = [\sigma][\tau][\sigma^{-1}] = [\sigma][\tau][\sigma]^{-1}$$

and so $\pi_1(G, e)$ is abelian. □

Problem 10.3.5

Problem 5 Notes

Solution

See the comments for problem 6 of the Summer 2012 exam (6.1.6).

Problem 10.3.6

Problem 6 Notes

Solution

[1] H is the composition of continuous maps, but we do not feel that it's relevant to decompose H entirely.

Remember the Five Lemma and the Long Exact Sequence axiom for homology. There isn't a lot to do here.

Topology Commentary

Fall 2013

Problem 10.4.1

Problem 1 Notes

Solution

The basic premise is to use the UMP of the group product to define the map and then use projections to show that the map is an isomorphism. It is possible to explicitly define each map being used, but that's unnecessary.

Problem 10.4.2

Problem 2 Notes

Solution

Not much to do except follow the definitions about.

Problem 10.4.3

Problem 3 Notes

Solution

This is a standard problem in algebraic topology and we have included the standard solution. Now, as a bonus, why do we use homology instead of homotopy groups?

Problem 10.4.4

Problem 4 Notes

Solution

Part (a) is, perhaps, the only surprising fact in this problem. Intuitively, it may seem that having a global frame on the whole manifold would induce a global frame on the boundary.

Problem 10.4.5

Problem 5 Notes

Solution

This is a very instructor-specific problem. As of late 2016, this is the only topology problem specifically relying on material covered in a specific iteration of algebraic topology.

Remark: Theorem 6.4.2 is proved (in Massey's book) using the Mayer–Vietoris exact sequence and the following lemma.

Lemma 10.4.1 (THEOREM 6.2, MASSEY PG. 211): *Let A be a subset of S^n which is homeomorphic to $[0,1]^k$, $0 \leq k \leq n$. Then $\widetilde{H}_q(S^n \setminus A) = 0$ for all $q \geq 0$.*

It is possible that this lemma may look very odd to you. If your course used simplicial homology rather than cubical homology (as Massey uses) then the idea of deleting cubes may seem unnatural. In particular, the proof of this lemma is a bit involved and really does (as presented by Massey) rely on the fact that he uses cubical homology. Of course they're the same homology groups so the same theorem holds for simplicial homology. So this is no big issue in terms of validity of the lemma, it just means that the presentation may be unclear if you just read that section of the book without having developed homology using cubes from the beginning. For a discussion of some various advantages and disadvantages to using cubes see this MathOverflow thread:

`http://mathoverflow.net/questions/3656/cubical-vs-simplicial-singular-homology`.

Problem 10.4.6

Problem 6 Notes

Solution

These are classical tricks for computing integrals. It has been a running joke that if you see an integral on the written exam, the answer is 0 because of Stokes' Theorem.[2] However, it has further been acknowledged that the "why" is trickier.

[2] This is frequently the case but that's not something to rely upon.

Topology Commentary

Summer 2014

Problem 10.5.1

Problem 1 Notes

Solution

This is a nice problem which uses the parametrized version of Stokes' Theorem instead of the full version. Otherwise it's just working through definitions.

Problem 10.5.2

Problem 2 Notes

Solution

The main trick to this problem is recognizing how to use the Implicit Function Theorem.

Problem 10.5.3

Problem 3 Notes

Solution

This problem can be viewed as one part of a larger theorem showing the equivalence of three definitions of orientation. A good resource for the "complete" proof is Warner's *Foundations of Differentiable Manifolds and Lie Groups*.

Problem 10.5.4

Problem 4 Notes

Solution

No real commentary for this problem – it's a standard problem and demonstrates a working knowledge of all of the Eilenberg–Steenrod axioms.

Problem 10.5.5

Problem 5 Notes

Solution

The essential trick to this problem is translating it into a problem about fundamental group and remembering that $\mathbb{R}^2$ is the universal cover of the torus.

Problem 10.5.6

Problem 6 Notes

Solution

See the comments for problem 6 on the Fall 2012 exam (6.2.6).

Topology Commentary

Fall 2014

Problem 10.6.1

Problem 1 Notes

Solution

The solution presented for this problem is almost certainly not optimal. In particular, Step 1 is lengthy and extremely computational; this is not the normal style of written qual problems. However, the expected results of Step 1 are *given* in the problem statement. Thus it may have been expected to mess around with the initial computation a bit and then move on to the later steps (which involve a little more understanding and a little less brutal computation).

Problem 10.6.2

Problem 2 Notes

Solution

See the comments for problem 5 of the Fall 2012 exam (10.2.5).

Problem 10.6.3

Problem 3 Notes

Solution

While cellular homology is a standard topic for algebraic topology, not every iteration covers the details of computing cellular homology. However, the idea of finding matrix representations of the maps and using Smith normal form to find the invariant factors is a powerful idea that ultimately bypasses many of those details.

Problem 10.6.4

Problem 4 Notes

Solution

This is a problem where the hint tells you everything. The suggestion of viewing Δf as a divergence suggests that the Divergence Theorem (for Riemannian manifolds) is going to come into play. Indeed, the solution only makes sense if you remember the statement of the Divergence Theorem and thus the pieces that you need to compute.

Problem 10.6.5

Problem 5 Notes

Solution

It may not be immediately apparent from the problem statement that we should use Brouwer degree to push our result through. However, the fact that it is a fixed-point theorem should suggest that such a trick is possible.

For the referenced results about degree, consult Hatcher. He certainly discusses when maps of spheres are homotopic to the identity or antipodal maps.

Problem 10.6.6

Problem 6 Notes

Solution

This problem can be viewed as one part of a larger theorem showing the equivalence of three definitions of orientation. A good resource for the "complete" proof is Warner's *Foundations of Differentiable Manifolds and Lie Groups*.

Topology Commentary

Summer 2015

Problem 10.7.1

Problem 1 Notes

Solution

See the comments for problem 5 on the Summer 2014 exam (10.5.5).

Problem 10.7.2

Problem 2 Notes

Solution

Other than remembering how local homology works (i.e., why it could be used to show that a space is *not* a topological manifold), the biggest trick to this problem is having the correct visualization of the spaces involved. While that's harder to definitively study, there are lots of topology textbooks that might give you some visualization practice (Hatcher, for instance).

Problem 10.7.3

Problem 3 Notes

Solution

See the comments for problem 6 on the Fall 2012 exam (6.2.6).

Problem 10.7.4

Problem 4 Notes

Solution

Remembering the formula for the Lie bracket of vector fields probably makes this computation faster, but that's somewhat unclear. The suggestion in the solution to compute $V(Wf)$ and $W(Vf)$, ignoring second-order terms, is probably better for qual stress (the problem breaks down further into smaller chunks that are more easily double-checked).

Problem 10.7.5

Problem 5 Notes

Solution

This problem basically checks what you remember about point-set topology and submersions. The only trick is remembering that $\mathbb{R}^k$ is connected (in the usual topology) and the consequences of that fact.

Problem 10.7.6

Problem 6 Notes

Solution

The first part of this problem is just having enough familiarity with all of the definitions to understand the objects defined in the problem statement. Then, transforming the problem to studying the kernel of $\omega - f^*\omega$ and a little diagram chasing gives the result.

Topology Commentary

Fall 2015

Problem 10.8.1

Problem 1 Notes

Solution

There is perhaps more exposition in the given solution than is actually required. After noting that f maps the 2-cell e^2 on $\mathbb{T}^2$ to the 2-cell on $\mathbb{S}^2$, it follows that $[f(e^2)]$ is the generator for $H_2(\mathbb{S}^2)$.

Problem 10.8.2

Problem 2 Notes

Solution

The essential part of problem is recognizing the fact that you want to use suspensions. Indeed, the fact that we are looking at Brouwer degree of a map (compared to its restriction to a smaller sphere) gives this away. After that, it's a matter of working through the details until the result pops out.

Problem 10.8.3

Problem 3 Notes

Solution

See the comments for problem 5 of the Fall 2012 exam (10.2.5).

Problem 10.8.4

Problem 4 Notes

Solution

This is a standard exercise in differential topology.

Problem 10.8.5

Problem 5 Notes

Solution

It is unclear what level of detail was required/expected for this problem. Conceptually, we take a nice vector field on the punctured sphere and argue that it must extend by 0 to the entire sphere. It is possible to go through and directly compute W on the punctured sphere, but this doesn't seem worthwhile.[3]

Problem 10.8.6

Problem 6 Notes

Solution

See the comments for problem 6 on the Fall 2013 exam (10.4.6).

[3]Indeed, even if you remember stereographic projection, are you really going to remember the pushforward by heart? If not, you've got a computation waiting for you.

Topology Commentary

Summer 2016

Problem 10.9.1

Problem 1 Notes

Solution

As stated on the actual exam, this problem omitted a somewhat essential detail: N must be connected. Take, for instance, $M = S^2$ and $N = S^2 \sqcup S^2$. Then M is compact and the inclusion of $M \hookrightarrow N$ is a submersion (actually, it is a local diffeomorphism at every point in M), but clearly not a covering map.

Problem 10.9.2

Problem 2 Notes

Solution

There are a couple ways to approach this problem. The first is the one detailed here: understand orientation in terms of differential forms and chug through the interior product computation.

The alternative approach is to view this as an algebraic topology problem, where we use Brouwer degree and understand "orientation-reversing" as having degree -1. The advantage to this approach is that, employing the correct theorems, the result falls out immediately. However, assuming that there should be an equal number of differential topology and algebraic topology problems, this "should" be solved using differential topology.

Problem 10.9.3

Problem 3 Notes

Solution

One of the biggest potential hurdles for solving this problem is knowing how to define g_{ij}. If you've spent some time with Riemannian metrics, it's natural to think of these coefficients as the entries of the Gram matrix for the Riemannian metric g. However, without that insight, you're rather stuck – try writing some true things down and hope for the best.

Problem 10.9.4

Problem 4 Notes

Solution

There isn't a lot going on in this problem except for definitions. Remembering the difference between a retraction and deformation retraction makes all the difference in the world. What can be said about i_* for A being a retraction of X?

Problem 10.9.5

Problem 5 Notes

Solution

Enjoy the visualization! Given the statement of the problem, it's unclear how much justification was expected. The provided solution went the verbal description route because this seemed more useful.

Problem 10.9.6

Problem 6 Notes

Solution

This cute little problem combines two of the most horrible things to see on a qualifying exam in the same place:

- A visually complicated setup to a problem.
- Explicitly working with the maps in the Mayer–Vietoris sequence.

That being said, if you're practicing for the topology exam, this is a beautiful problem.

Topology Commentary

Fall 2016

Problem 10.10.1

Problem 1 Notes

Solution

While more elegant, the first solution presented is employing a major theorem that isn't typically proved in the differential topology course. Thus it may be better to focus on the second solution instead.

Problem 10.10.2

Problem 2 Notes

Solution

This problem is rather cute and is mostly about the Leibniz rule and properties of the differential. Figuring out how to rewrite an element of the kernel as a sum of products might be a little tricky, but there aren't a lot of choices if you're going to make use of the hint.

Problem 10.10.3

Problem 3 Notes

Solution

In standard "compute the integral" fashion, this solution relies on making the correct observation and then applying Stokes' Theorem.

Alternate solution: For integration on (compact) product manifolds, we have a Fubini-type theorem and so

$$\int_{T^2} z dx \wedge dt = \left(\int_{S^1} dx \right) \left(\int_{S^1} z dt \right).$$

We could compute this, using the line integrals and the information we have about orientations.

Problem 10.10.4

Problem 4 Notes

Solution

The part of this problem most likely to trip you up is the difference between homotopy and isotopy. The curve A is not isotopic to a circle – no self-intersections are allowed in an isotopy. However, for a homotopy, no such restrictions apply.

Problem 10.10.5

Problem 5 Notes

Solution

Once you've sorted out the CW structure for this space, it's just a matter of working through the computations.

Problem 10.10.6

Problem 6 Notes

Solution

If you know the correct definitions for this problem, it's actually a combinatorics problem rather than something topological. This may cause jumping for joy or it may be a source of bemusement.

Topology Commentary

Summer 2017

Problem 10.11.1

Problem 1 Notes

Solution This is a classic problem and there probably as many ways to solve it as there are topologists on the planet. However, none of the proofs are particularly long[4] and the key idea is that this is really a statement about finite dimensional vector spaces.

Problem 10.11.2

Problem 2 Notes

Solution

One of Vlad's favorite problem types. Depending on the specific material covered in your version of the course, some of these arguments may not seem familiar. However, there isn't anything particularly deep going on, except in part (f), where the Poincaré–Hopf Theorem makes an appearance. Compare to (6.1.2) and (6.4.2) for other examples.

Problem 10.11.3

Problem 3 Notes

Solution Reading the problems should put Stoke's Theorem immediately into your mind, so the question becomes how to use the problem statement to set up a tractable integral. Luckily, all of the pieces are there and, as long as you remember the various pieces of integration algebra, everything comes together nicely.

Problem 10.11.4

Problem 4 Notes

Solution

Another reason to be wary of topological structures with torsion.

Problem 10.11.5

Problem 5 Notes

Solution

Just to be clear, there isn't anything magical about 17 (73 would have worked as well). To see why you might have ended up with this decomposition, it is helpful to write out the Mayer–Vietoris sequence first to identify where the arrow between $\widetilde{H}_k(S^n)$ and $\widetilde{H}_{k-1}(S^{n-1})$ appears and then determine what you need to be true about the nearby components to derive the isomorphism.

[4]Depending on the assumed background...

Problem 10.11.6

Problem 6 Notes

Solution

The version of 124 this particular year didn't make use of the Smith Normal Form explicitly, but it is a convenient tool to know and does offer a general solution method for these types of problems. Also, the images and kernels were clear without relying on SNF anyway.

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Applied Commentary

Applied Commentary

Summer 2017

Problem 11.1.1

Problem 1 Notes

Solution

Iterative methods were a significant topic in 116 the year of this exam. Notice that this problem really just relies on being able to make use of (and interpret) the (very standard) Taylor expansion. Hopefully by the time you are taking this exam you have done similar things dozens of times.

Problem 11.1.2

Problem 2 Notes

Solution

Straightforward basics of numerical linear algebra. Note that for parts (2) and (3) there is no need to write out the matrix multiplications that make up this procedure but if the topic were something like Householder reflections some more detail might be necessary. No need to exposit too much in the explanation of (4) although there is obviously more the $c$ould be said.

Problem 11.1.3

Problem 3 Notes

Solution

Another Taylor problem, really this is just a check of whether you are familiar with the definitions from the last third of the course. The method for determining the region for part (3) was covered on homework in the class.

Problem 11.1.4

Problem 4 Notes

Solution

Nothing too exciting happening here, mostly just checking the basics of generating functions and Markov processes. Note that this problem was an actual homework problem from 106.

Problem 11.1.5

Problem 5 Notes

Solution

The first part has a large number of steps but the methods and overall outline are very familiar, particularly in the homogeneous case. Note that this problem was an actual homework problem from 106.

Problem 11.1.6

Problem 6 Notes

Solution

A little random variable theory and some more checking in on basic definitions. Note that this problem was an actual homework problem from 106.

Appendices

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Appendix A: Qual Course Instructors

Here is the list of faculty members who have taught the qual courses since the written qual system was introduced. Each faculty member has their own preferences and approaches to the material so it is useful to consider who wrote each question. In 2016, Math 111 was taught by a postdoc, Sam Miner, who left before the written exam. The Galois questions that year were written by Tom Shemanske. The applied mathematics courses were not offered before 2017.

Qual Course Professors							
Course	2012	2013	2014	2015	2016	2017	2018
101	Webb	Webb	Williams	Shemanske	Webb	Voight	Voight
111	Shemanske	Shemanske	Shemanske	Voight	Shemanske*	Shemanske	Shemanske
103	Williams	Gordon	Gordon	Williams	Gordon	Clare	Williams
113	Trout	Trout	Trout	Clare	Williams	Williams	Trout
124	Chernov	Chernov	Webb	Chernov	Chernov	Chernov	Chernov
114	Arkowitz	Williams	Webb	Webb	Sadykov	van Erp	Webb
106	n/a	n/a	n/a	n/a	n/a	Fu	Fu
116	n/a	n/a	n/a	n/a	n/a	Gelb	Gelb
126	n/a	n/a	n/a	n/a	n/a	n/a	Kim

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Appendix B: Textbook Recommendations

Finding the right source to study the material for the qualifying exam can be a challenging task. No one textbook contains all of the necessary material, matches individual preferences about desired level of detail, formalism, and abstraction. This makes it impossible to recommend the "best" textbook for any particular subject. That being said, this appendix contains some thoughts collected from current graduate students about resources that may be helpful to you. No matter your favorite textbook, we recommend that you expose yourself to a variety of sources and perspectives in each subject.

Math 101

The many different components of this course tend to lend themselves to different textbooks:

- The basic Sylow theory is covered well in Dummit and Foote or any other standard undergraduate text.
- Grove's *Algebra* is a great source for material about module theory, particularly in the setting of linear algebra and the fundamental theorem.
- Jacobson's *Basic Algebra I* and *II* are excellent overall sources for the class.

Math 111

In most renditions of this course, the course starts with ring theory and then transitions into Galois theory. The ring theory is found in standard algebra references (see above). It is probably worthwhile to flip through several approaches to some of the fundamental constructions like the polynomial ring and finite field extensions. For Galois theory, Lang's *Algebra* is a fantastic resource. His chapter on Galois theory contains everything you need to know.

Math 103

Measure theory: Royden and Fitzpatrick or Folland are both solid sources for this material.

Complex Analysis: Get a copy of Ullrich's *Complex Made Simple* – it is a fantastic and intuitive book that covers the main results of the course in a very readable fashion. Conway's *Functions of One Complex Variable* is another (less conversational) source that is still more readable than Rudin.

Math 113

The chosen textbook for the course is usually a good enough source. *Fourier Analysis* by Stein and Shakarchi is a nice reference for that portion of that course.

Math 104

The standard reference for differential topology is *An Introduction to Smooth Manifolds* by John Lee. However, in case you want more references, you can consult any of the texts below:

- Warner's *Foundation of Differentiable Manifolds and Lie Groups* is indispensable.
- Big Spivak I has a proof of the equivalence of the various tangent bundle definitions that is worth looking through.
- Guilleman and Pollack and Boothby are both useful supplemental sources for this course.

While not an official textbook, it's also worthwhile finding a copy of David Webb's notes on multivariable analysis. This is a fantastic introduction to multivariable calculus that underlies all of most versions of Math 104.

Math 114

Oddly enough, there seems to be the most division over the best resources for Math 114. Below is a substantial collection of textbooks and hopefully one of them will meet your needs:

- Hatcher is the "standard" source. Unfortunately, while most people agree that the pictures are wonderful, the organization doesn't fit well into a quarter-long class (leading to significant jumping around).
- Massey is another favorite. Some people find him too conversational, but that will depend on your preferences anyway.
- Rotman's *An Introduction to Algebraic Topology* has a wonderful treatment of homology groups that includes all the wonderful category theory that it ought to have.
- Fold is another useful resource, depending on the instructor.

Math 106 and 116

We don't know yet.

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Appendix C: Written Qual Question Topics

2012 Summer					
101	111	103	113	124	114
Tensor Natural Isomorphism	Ideals in Quotients of Polynomials	Definitions of Continuity	Sequences of Functions	Orientability of Manifolds	Covering Maps
Matrices, Bases Diagonalizability	Galois Group $x^9 - 8$	Holomorphic/ Analytic	Adjoints iff	Function Degree on Manifolds	Homology of Torus
Semidirect Product	Field Extensions Irreducibility	Cauchy– Riemann	Bounded Operators	Manifold Charts	Wedge/ Poducts

2012 Fall					
101	111	103	113	124	114
Sylow/ Solvability	Three Conditions Normality	Residue Theorem	Hahn– Banach	Smooth Tensor Fields	Covering Maps
Quotients Cyclic Groups	Galois Group $x^{15} - 8$	Holomorphic/ Analytic	Radius of Convergence	Critical Point Matrix Calculus	Eilenberg– Steenrod
Projective/Flat Modules	Extension Degree Embedding in S_n	Measures Borel Sets	Uniform Continuity	Stokes' Theorem	Cellular Homology

2013 Summer					
101	111	103	113	124	114
Sylow/ p–groups	Ideals in Quotients of Polynomials	$\limsup$ Definitions	Sup Norm of Functions	Orientability of Manifolds	Fundamental Group
$k[x]$– Modules	Algebraic Distinguished	Holomorphic/ Analytic	Orthogonal Complements	Manifold Embeddings	De Rham Cohomology
Dual Modules Tensors	FTGT Norms	Integral Measures	Bounded Operators	Lie Bracket Vector Fields	Wedge and Cup Poducts

2013 Fall					
101	111	103	113	124	114
Diagonalizable Complexification	Irreducibles UFD, PID	Anti–derivatives on $\mathbb{C}$	Convolution Products	Equivalence of Smoothness	No Retraction Fixed Point
Matrices, Bases Bilinear Form	Galois Group Ugly Product	Entire Bounded	Pythagorean Theorem	Parllelizable Manifolds	Homology of a Product
Groups S_n	Frobenius Perfect Fields	Limits/Integral Theorems	Bounded Operators	Stokes' Theorem	De Rham Cohomology

2014 Summer					
101	111	103	113	124	114
Projective Modules	Types/Degrees Field Extensions	Measurable Functions	Direct Product Banach Spaces	Integration on Manifolds	E & S Coproduct
Canonical Forms Nilpotent	Galois Group Properties	Measures Integrations	Self–Adjoint Operators	Inverse Function Theorem	Homotopy Products
Sylow Theorems	Noetherian Factorization	Entire Functions	Hahn– Banach	Orientability on Manifolds	CW Complex

2014 Fall					
101	111	103	113	124	114
Canonical Forms	Explicit Galois Groups	DCT/ MCT	Closed Graph Theorem	Cross Product Manifold	Lifts of Covering Maps
Free Modules Tensor Product	Algebraic Elements	L^p Spaces	Completeness supNnorms	Volume Form on Manifolds	Hairy Ball Theorem
Sylow Theorems	Roots of Unity Finite Fields	Complex Integrals	Finite Rank Operators	Orientability on Manifolds	CW Complex

2015 Summer					
101	111	103	113	124	114
Module of Fractions	Complex Field Extensions	Analytic Unif. Convergence	(un)Bounded Functionals	Compute Lie Bracket	Nullhomotopic Sphere Maps
Projective Modules	Finite Field Homomorphisms	Complex Asymptotics	Banach Algebras	Compact Submersions	Local Homology
$GL_n(\mathbb{F}_p)$ Sylow	Compute Galois Group	Lebesgue Measure	Adjoint Properties	Differential Form Algebra	CW Complex

2015 Fall					
101	111	103	113	124	114
Free Module Decompositions	Irreducible Polynomials	Analytic Convergence	Banach Spaces	Submersion Properties	Nullhomotopic Sphere Maps
Simple Groups	Compute Galois Splitting Field	Singularity Properties	Weak Star Topology	Hairy Ball Theorem	Degree of Sphere Functions
Group Commutators	Quadratic Extensions	Product Measure	Shift Map Properties	Integration on Manifolds	Lifts of Covering Maps

2016 Summer					
101	111	103	113	124	114
Canonical Forms Complexification	Polynomial Rings	Integration of Sequences	Weak Convergence	Submersion Covering	Deformation Retractions
Center of p–group	Algebraic Extensions	Product Measure Tonelli	Integral Operators	Sphere Function Degree	Universal Covers
Group Tensor Products	Lattice of Subgroups	Entire Functions	Function Convergence	Induced Metrics	Mayer–Vietoris Computation

2016 Fall					
101	111	103	113	124	114
Projection Operators	UFD Properties	Integration and Measures	Bounded iff Continuous	$\mathrm{SL}(n, \mathbb{R})$ Manifold Definition	Homotopy and Retractions
$\mathrm{SL}(3, \mathbb{F}_3)$ Sylow	Splitting Fields	Measurable Sets	Bounded Adjoints	Tangent Bundle Isomorphism	Homology Computation
Direct Sum Decomposition	Galois Group Properties	Cauchy Integral	Holder Compactness	Integration on Manifolds	CW – Euler Characteristic

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Appendix D: Qual Course Syllabi

Math 101 Syllabus
Standard Text: Dummit and Foote: Abstract Algebra, Chapters 4, 5, 10, 11, 12

1. [4 days] Basic Linear Algebra:

   (a) (Assumed) Linear independence, span, basis, dimension, independent sets extend to a basis, generating sets can be pared down to a basis.

   (b) Coordinates and matrix of a linear transformation relative to a basis, change of basis. Examples: projection onto a hyperplane, rotations.

   (c) Row reduction, echelon form, and consequences: free variables, pivot variables, kernel and image, rank-nullity theorem for $T : \mathbb{R}^n \to \mathbb{R}^m$ (via free and pivot variables), elementary row operations and invertibility. Parallel comments for column operations. Given $A \in M_{m\times n}(F)$, discuss representative of cosets $GL_m(F)A$, $AGL_n(F)$, and $GL_m(F)AGL_n(F)$, the last as precursor to Smith normal form.

   (d) Rank - Nullity (vector space form)

   (e) Foreshadow Smith normal form by considering $A \in M_{m\times n}(\mathbb{Z})$ and row and column operations (over $\mathbb{Z}$) to produce the nice representative in $GL_m(\mathbb{Z})AGL_n(\mathbb{Z})$ (when $m = n$, $\mathrm{diag}(d_1, \ldots, d_n)$ with $d_i \in \mathbb{Z}$ and $d_i \mid d_{i+1}$, $1 \le i \le n-1$). Example: structure of $\mathbb{Z}^n/K$ where K is a subgroup generated by a collection of vectors. Interpret as linear map and use two sided equivalence to produce a new basis so that $\mathbb{Z}^n/K \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z}$ ($d_i \mid d_{i+1}$) [foreshadowing invariant factor theorem].

2. [4 days] Modules: basic properties.

   (a) Definitions, examples (vector spaces, abelian groups, $T : V \to V$ linear map to $k[x]$-module structure on V. Notion of a k-algebra ($M_n(k)$, $k[x]$, $End_k(V)$, $k[G]$) and UMP: given any k-algebra A and $a \in A$ there is a unique k-algebra map $k[x] \to A$ taking $x \mapsto a$.

   (b) Direct sums of modules (external and internal); spin off internal direct product of groups. Discuss product and direct sum of vector spaces, mapping properties. Define product and coproduct of modules and their construction. Show $\mathrm{Hom}_R(N, \prod M_\alpha) \cong \prod_\alpha \mathrm{Hom}_R(N, M_\alpha)$, $\mathrm{Hom}_R(\coprod_\alpha M_\alpha, N) \cong \prod_\alpha \mathrm{Hom}_R(M_\alpha, N)$ and $End(k^n) = \mathrm{Hom}(k^n, k^n) \cong M_n(End_k(k)) \cong M_n(k)$

3. [3 days] Exact sequences of modules; split exact sequences via sections or retractions (existence of section equivalent to existence of a retraction). Free modules and their construction; Short exact sequences with $0 \to N \to M \to F \to 0$ with F free split. Any R-module is the quotient of a free R-module (review isomorphism theorems if needed). Localization of modules, connection to exactness, action on direct sums; application: rank of a module over an integral domain is the dimension of the localization over the field of fractions.

4. [6 days] PIDs; Finitely generated modules over PIDs, invariant factor and elementary divisor theorems, applications to rational and Jordan canonical forms. Diagonalizability.

5. [1 day] Dual Modules (duality and free modules)

6. [2 days] Sesquilinear forms. Unitary, Hermitian operators, unitary diagonalization. Real symmetric matrices and spectral theorem.

7. [8 days] Group actions, G-set structure theorem, class equation, p-groups symmetric group, conjugacy classes in S_n, Sylow theorems, semidirect products and split extensions, classifying groups of small orders.

Optional topics:

1. [2 days] (optional) Bilinear forms, isometry groups, connections to dual spaces.

Math 111 Syllabus (cross-listed with Math 81)
based on Lang, *Algebra*

1. [3 days: II.2, II.4 - II.5] Commutative rings, prime and maximal ideals, CRT, evaluation and reduction homomorphisms, Localization of rings (field of fractions), irreducibles, primes, UFDs, PIDs, Euclidean domains.
2. [3 days: IV.1 - IV.3] Polynomials in one variable, over UFDs, Gauss's lemma, irreducibility criteria.
3. [3 days: V.1] Finite and Algebraic Field Extensions.
4. [3 days: V.2 - V.3] Splitting fields, normal extensions, and algebraic closures; uniqueness.
5. [4 days: V.4 - V.5] Separable extensions, primitive element theorem, Finite fields.
6. [1 day: V.6] Inseparability (intro only)
7. [4 days: VI.1]Galois Extensions: Fundamental theorem, composite extensions
8. [3 days: VI.2] Galois groups of polynomials.
9. [3 days: VI.3] Cyclotomic extensions and polynomials

Optional topics: group rings, polynomial rings in several variables, compass and straightedge constructions, solvability by radicals, infinite Galois groups

MATH 103 SYLLABUS (CROSS-LISTED WITH MATH 73)

1. Abstract Measure Theory (12 Lectures)
   (a) Measures, σ-algebras and all that.
   (b) An Example: Lebesgue measure on $\mathbf{R}$ and/or $\mathbf{R}^n$.
   (c) Integration in an abstract measure space
   (d) The convergence Theorems and applications.
   (e) Product measures, Tonelli and Fubini.
   (f) Sources
      i. We have in mind cherry picking from Rudin's *Real & Complex* (Chapters 2, 6 and 8) since that will be the usual reference for the second part of the course. Instructors will have to develop Lebesgue measure on their own or possibly using Royden & Fitzpatrik as a guide.
      ii. Obviously, time constraints and the instructor's interests will dictate what topics can be covered and at what depth. The topologists would love some attention paid to Lebesgue measure in $\mathbf{R}^n$ at some point.
2. Complex Analysis (15 Lectures)
   (a) Elementary Properties
      i. Complex differentiation, Cauchy Riemann equations and path integrals
      ii. Local Cauchy Theorem
      iii. Holomorphic implies analytic
      iv. Global Cauchy Theorem
      v. Sources
         A. The basic source we have in mind is Chapter 10 of Rudin's *Real & Complex*. This can be followed fairly closely — even if it is fairly sophisticated.
         B. Dana was taught that in an outline, there had to always be at least two sub-parts under any given item.
   (b) Selected Topics — Lecturer's Discretion
      i. Maximum Modulus
      ii. Isolated Singularities and Laurent Series
      iii. Residue Theorem and Applications
      iv. Argument Principle and Roche's Theorem
      v. Normal Families and Riemann Mapping Theorem
      vi. Sources
         A. When Dana tried this before, he picked and chose from Chapter's 12–14 of Rudin's *Real & Complex*.
         B. Sample Goal: try to build up enough background to at least pretend to prove Theorem 13.11 (Rudin) (at least $(b) \iff (c) \iff (d) \iff (f)$). That can't be done without leaving the proofs of some of the harder results.

Math 113 Syllabus

1. Banach Spaces and Hilbert Spaces (12 Lectures)
    (a) Inner products and linear functionals
    (b) Orthogonal sets, Bessel's inequality and Parceval.
    (c) General Banach Spaces
    (d) Consequences of Baire's Theorem (Open mapping, Closed Graph and Principle of Uniform boundedness).
    (e) Hahn-Banach and applications.
    (f) Sources
        i. This section is the most flexible.
        ii. A minimal approach could be crafted out of Rudin's Real & Complex Chapters 4 and 5.
        iii. A more thorough treatment could be excised from Royden/Fitzpatrick Chap. 13, §§14.1–2 and §§16.1–5.
2. General Fourier Series (7 Lectures)
    (a) Motivations: Vibrating strings and the Heat equation
    (b) Basic Fourier Series and Uniqueness
    (c) Convolutions and good kernels
    (d) Cesaro and Abel summability
    (e) Convergence issues
    (f) Applications: Heat equation on the circle, Weyl's equidistribution theorem, isoperimetric inequality, etc.
    (g) Sources
        i. Chapters 1 – 4 of Stein and Shakarchi: "Fourier Analysis: An Introduction".
3. The Fourier Transform on the Real Line (8 Lectures)
    (a) Definition and Schwartz space
    (b) Fourier inversion
    (c) Plancherel formula
    (d) Extensions to functions of moderate decrease
    (e) Weierstrass approximation theorem
    (f) Application to heat equation
    (g) Poisson summation formula
    (h) Source
        i. Chapter 5 of Stein and Shakarchi: "Fourier Analysis: An Introduction"

Math 104 Syllabus

Review of differential calculus in $\mathbb{R}^n$
The derivative of a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, C^1 implies differentiable, the Jacobian matrix, the chain rule, the inverse and implicit function theorems, etc.; many of these topics may be sketched or reviewed without proof.

Smooth manifolds
The definition of a smooth manifold, coordinate charts, the tangent space and the ways of defining tangent vectors, the derivative of a smooth map of manifolds, smooth vector fields, etc.

Multilinear alternating algebra Tensors, alternating tensors, the wedge product and exterior algebra, behavior of tensors under linear maps, orientation of a vector space.

Differential forms
Differential forms, the exterior derivative, the Poincaré Lemma, orientation of a manifold.

Brief review of integration of functions on $\mathbb{R}^n$
A brief review of definitions, Fubini's Theorem.

Integration of differential forms
Parametrized integral of a k-form over a k-chain, smooth partitions of unity, unparametrized integral of an n-form with compact support on an oriented smooth n-manifold.

Stokes's Theorem
The modern Stokes's Theorem $\int_M d\omega = \int_{\partial M} \omega$ for the integral of an exact n-form on an oriented n-manifold with boundary, the classical integral theorems of vector calculus as special cases of the modern theorem.

Bibliography

W. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, second edition. Pure and Applied Mathematics **120**. Academic Press, Inc., Orlando, FL, 1986.

M. Spivak, *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*, W. A. Benjamin, Inc., New York-Amsterdam, 1965.

Math 114 (cross-listed with Math 74) - Algebraic Topology

3-4 weeks on the fundamental group and covering spaces.
6-7 weeks on homology theory

The Fundamental Groups and Covering Spaces

1. Homotopy, definition of π_1 and its basic properties, van Kampen Theorem
2. Covering spaces: lifting properties, the Galois correspondance, deck transformations, the universal cover
3. Applications and examples

Homology Theory

1. Basic homological algebra: chain complexes, exact sequences, chain maps, chain homotopy
2. Construction of the singular homology of a pair of spaces
3. Eilenberg-Steenrod axioms: excision, exactness, homotopy
4. Computational techniques (Mayer-Vietoris sequence,..)
5. Applications (Brouwer fixed point theorem, hairy ball theorem, ..)