STAIR CLIMBING PROBLEMS

CROSSROADS ACADEMY
MATHCOUNTS PREPARATION

I) How many ways can Joe climb 10 stairs if he can take them 1, 2, or 5 at a time? What if there are 16 stairs? 50 stairs?

Answer 1. *The purpose of this problem is a reminder about simplifying counting with recurrence relations. Problems of this sort frequently occur in the sprint and target rounds of Mathcounts competitions. The idea is that while it is certainly possible to figure out the number of ways by cases for* $n = 10$ *or* $n = 16$ *it would be easy to make small mistakes or miscount even on problems of that size. On the other hand, constructing a recurrence relation only requires that we count the smallest cases by hand and then get all of the higher numbers for free.*

To construct our relation, let's let a_n *be the number of ways to climb* n *stairs taking 1, 2, or 5 at a time. We have to accomplish two steps: first, we need to compute the initial conditions,* a_1, a_2, a_3, a_4 *and* a_5*, then we can compute the higher cases by observing that* $a_n = a_{n-1} + a_{n-2} + a_{n-5}$ *since we can simply remove the last step that we take from any pattern of 1, 2, or 5 that takes us to the* n^{th} *step. We can write out the values for the initial conditions by hand:*

$$
\begin{aligned}
a_1 = \ & 1 = & \{(1)\} \\
a_2 = \ & 2 = & \{(1+1),(2)\} \\
a_3 = \ & 3 = & \{(1+1+1),(1+2),(2+1)\} \\
a_4 = \ & 5 = & \{(1+1+1+1),(1+1+2),(1+2+1),(2+1+1),(2+2)\} \\
a_5 = \ & 9 = & \{(1+1+1+1+1),(1+1+1+2),(1+1+2+1),(1+2+1+1), \\
& & (2+1+1+1),(1+2+2),(2+1+2),(2+2+1),(5)\}
\end{aligned}
$$

Now that we have these initial conditions we can compute the number we need using our recurrence relation:

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\begin{aligned}
a_6 &= a_5 + a_4 + a_1 = & 9+5+1 = 15 \\
a_7 &= a_6 + a_5 + a_2 = & 15+9+2 = 26 \\
a_8 &= a_7 + a_6 + a_3 = & 26+15+3 = 44 \\
a_9 &= a_8 + a_7 + a_4 = & 44+26+5 = 75 \\
a_{10} &= a_9 + a_8 + a_5 = & 75+44+9 = 128 \\
\vdots &= \vdots = & \vdots \\
a_{15} &= a_{14} + a_{13} + a_{10} = & 3,142 \\
\vdots &= \vdots = & \vdots \\
a_{50} &= a_{49} + a_{48} + a_{45} = & 237,139,442,616
\end{aligned}
$$

II) How many ways can Sarah climb 15 stairs if she can only take them 2 or 6 at a time?

Answer 2. *Although it is tempting to try and repeat our successful method from the previous problem here, that would be overkill. The key here is to observe that we can only reach even numbered stairs taking steps of size two and six. Thus, there is no way for Sarah to do this and the answer is zero. More generally, if we are allowed to climb stairs* $\{m_1, m_2, \ldots, m_k\}$ *at a time, then we can't reach any stair who number is not a multiple of* $\gcd(m_1, m_2, \ldots, m_k)$*. In the previous problem, we didn't have to worry, since* $\gcd(1,2,5) = 1$*, but here* $\gcd(2,6) = 2$ *and the fifteenth stair is out of reach.*

III) How many ways are there to divide 10 doughnuts among 4 students if everyone must get at least one doughnut? What if there are 15 doughnuts? or 7 people? or n doughnuts and k people?

Answer 3. *This is an example of a famous problem/solution method known as "stars–and–bars". Since the doughnuts are indistinguishable but the students are distinguishable, our problem reduces to deciding how many doughnuts each student receives. We can imagine lining the doughnuts up in a row and separating them into 4 collections, one for each student. We can represent this diagrammatically with the doughnuts as* $*$*'s and the diving lines by* $|$*. For example,* $** \mid *** \mid ****|*$ *represent the first student getting 2 doughnuts, the second student gets 3 doughnuts, the third students get 4 doughnuts, and the fourth student gets 1 doughnut.*

Since each student must get at least one doughnut, any permissible distribution of the doughnuts can be represented by choosing 3 of the 9 spaces between the doughnuts to place the bars. Thus, the solution is $\binom{9}{3} = 84$*. If we have 15 doughnuts then there are 14 places to place the three bars or* $\binom{14}{3} = 364$*, if there are 7 people, we need to place 6 bars in the 9 spaces:* $\binom{9}{6} = 84$*. In general, if there are* n *doughnuts and* k *people there are* $n-1$ *places between the doughnuts to place the lines and* $k-1$ *lines that need to be placed giving* $\binom{n-1}{k-1}$ *total permissible distributions.*

IV) How many ways are there to divide 10 doughnuts among 4 students if some people are allowed to get zero? What if there are 15 doughnuts? or 7 people? or n doughnuts and k people?

Answer 4. *Here we can apply the same type of reasoning as in the previous problem, but there are some distinguishing features. We still want an arrangement of* n *doughnuts and* $k-1$ *bars, but we can have several bars between the same two doughnuts so counting the distributions with combinations requires a different approach than the previous problem. Notice that there are* $n+k-1$ *total objects that we want to arrange and any possible arrangement of the doughnuts and bars is permissible. Thus, there are* $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$ *total arrangements since we can describe any arrangement by either choosing the positions of the doughnuts or the bars. Note that this logic is similar to the reasoning for the existence of* $\binom{m+n}{n} = \binom{m+n}{m}$ *diagonal paths on an* $m \times n$ *grid that only go up or right.*

V) Imagine that we form a path connecting the positive integers, starting at 2, where two integers are connected if the smaller one divides the larger? How many steps does it take to get from 435 to 67891340? What if we only connect two integers if the smaller one is prime and divides the larger one?

Answer 5. *Since 5 divides both 435 and 67891340 in both situations we can find a path of length 2:* $435 \to 5 \to 67891340$.

The more interesting question here is the following: what is the maximum length of any shortest path between any pair of positive integers m *and* n*. In the first setting the maximum length is 2 since we can always take the following path:* $m \to mn \to n$*. In the second setting the maximum length is 4 since we can always take primes* p *and* q *so that* p *divides* m *and* q *divides* n*. Then, the path* $m \to p \to pq \to q \to n$ *always works. Some pairs may have a shorter path in* m *or* n *is prime or if, as in the case of 435 and 67891340,* m *and* n *share a common factor.*

1. Doubly Indexed Recurrences

VI) What is $\binom{n-1}{k} + \binom{n-1}{k-1}$?

Answer 6. $\binom{n}{k}$*. This is the classic binomial identity that defines the construction of Pascal's triangle. A combinatorial interpretation of this identity can be given using the interpretation that* $\binom{n}{k}$ *is the number of ways to choose a committee of* k *people from a collection of* n *people. Lining the people up in a row, we can separate the committee selections into two categories: those that contain the last person and those that don't. There are* $\binom{n-1}{k-1}$ *committees that contain the last person since we need to choose* $k-1$ *people from the remaining* $n-1$ *and* $\binom{n-1}{k}$ *committees that do not contain the last person since we still need to choose all* k *members from the remaining* $n-1$*people. .*

VII) How many ways are there to write 8 as a sum of 3 positive integers (order doesn't matter)? What about 9? What if we are allowed to use 4 integers? or n int a sum of k parts?

Answer 7. *Sums of these forms where an integer is written as an unordered sum of positive integers in known as a partition. The total number of partitions of* n *is frequently denoted* $p(n)$*, while the number of partitions into* k *summands is denoted* $p(n,k)$*. Thus, the original problem is denoted by* $p(8,3) = 5$ *since the partitions are* $\{(1+1+6), (1+2+5), (1+3+4), (2+2+4), (2+3+3)\}$.

Like the binomial coefficents, we can form a 2-d array for $p(n,k)$ *using a recurrence relation. The idea here is that we can separate the partitions into two cases: either* *there is a 1 in the representation or there is not. If there is a 1 in the representation we can remove the 1 from a sum to obtain a partition of $n-1$ into $k-1$ summands. If there is not a 1 in the representation we can subtract 1 from each term in the sum to obtain a representation of $n-k$ into k summands. Thus, our recurrence relation is $p(n,k) = p(n-1,k-1) + p(n-k,k)$.*

VIII) Four people are standing in a line. In how many ways can they rearrange themselves so that no one is standing in the same spot that they were originally?

Answer 8. *Permutations where no elements are sent to their original position are known as derangements. There are 9 derangments of length 4: $\{(2143),(2341),(2413),(3142),(3412),(3421),(4123),(4312),(4321)\}$. There isn't a particularly convenient closed form formula for the general number of derangements (although there are several interesting sum formulae relating derangments to e like $d_n = n!\sum_{i=0}^{n} \frac{(-1)^i}{i!}$) but they do satisfy a recurrence relation. Let d_n be the number of derangements of length n.*

Let p be an arbitrary derangement of length n. Then p sends n to some other integer j between 1 and $n-1$. There are two possibilities either p sends j to n or it doesn't. If p sends j to n then the remaining integers form a derangement of length d_{n-2}. If j is not sent to n then it is sent to some other integer i. We can swap where p sends n and where p sends j (so n goes to i and j goes to j) to obtain a derangement on the $n-1$ integers not equal to j. This gives us the formula: $d_n = (n-1)(d_{n-2} + d_{n-1})$. Since $d_1 = 0$ and $d_2 = 1$ we can compute all the rest of the d_n using this recurrence.

IX) n people standing in a line rearrange themselves into a new random order. What is the probability that no one is standing in the same spot that they were in originally?

Answer 9. *This is equal to the number of derangements of length n divided by the number of permutations of length $n = n!$. In the limit, this approaches $\frac{1}{e} \sim .368$.*