NUMERICAL SQUAREROOTS

Sometimes, we need to know the squareroot of an integer for a problem when the integer isn't a perfect square. We could make an estimate based on nearby squares, for example we know that $4 < \sqrt{20} < 5$ since $4^2 = 16$ and $5^2 = 25$, but this is less useful as the numbers get larger. Luckily, there is a fairly straightforward method for numerically determining the squareroot known as Newton's method. This is an **iterative** method which means that we perform a series of steps, over and over again, until we get an answer that is close enough to the value that we want to be satisfactory.

Let's say that we want to find $\sqrt{x}$. Newton's method starts with a "guess" or a number that we know is at least close to the actual answer. Usually, we can take $\lfloor\sqrt{x}\rfloor$ to be our starting point, so we will call this number $\lfloor\sqrt{x}\rfloor = x_0$. Then, we repeatedly update this number using the following rule:

$$x_n = x_{n-1} - \frac{x_{n-1}^2 - x}{2 \cdot x_{n-1}}$$

For each time through the process, the value of x_n gets closer and closer to $\sqrt{x}$. In other words $|x_n - \sqrt{x}| < |x_{n-1} - \sqrt{x}|$. A visualization of this process can be found here: https://www.math.dartmouth.edu/∼ddeford/sage4.html.

Example: Let's say we want to find $\sqrt{20}$ so we set $x = 20$. Then $x_0 = \lfloor\sqrt{x}\rfloor = \lfloor\sqrt{20}\rfloor = 4$. Now, we update this value to get better and better approximations:

$$\begin{aligned}
x_1 = &\quad x_0 - \frac{x_0^2 - x}{2 \cdot x_0} \\
x_1 = &\quad 4 - \frac{16 - 20}{2 \cdot 4} \\
x_1 = &\quad 4 - \frac{-4}{8} \\
x_1 = &\quad 4.5
\end{aligned}$$

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\begin{aligned}
x_2 &= x_1 - \frac{x_1^2 - x}{2 \cdot x_1} \\
x_2 &= 4.5 - \frac{4.5^2 - x}{2 \cdot 4.5} \\
x_2 &= 4.5 - \frac{.25}{9} \\
x_2 &= \frac{161}{36} \\
x_3 &= x_2 - \frac{x_2^2 - x}{2 \cdot x_2} \\
x_3 &= \frac{161}{36} - \frac{\frac{161}{36} - 20}{2 \cdot \frac{161}{36}} \\
x_3 &= \frac{51841}{11592} x_3 \approx 4.4721...
\end{aligned}
$$

If we square this final value we get 2687489281/134374464 which is 20.000000007441890, not too bad for only three steps through the method.

Problem: Compute an approximation for $\sqrt{50}$ using this method.