LAGRANGE INTERPOLATION

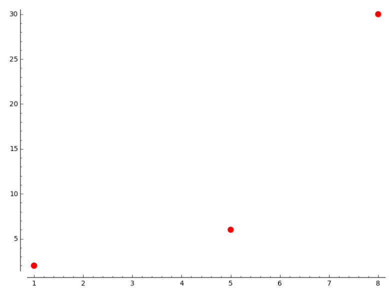
DARYL DEFORD

1. Introduction

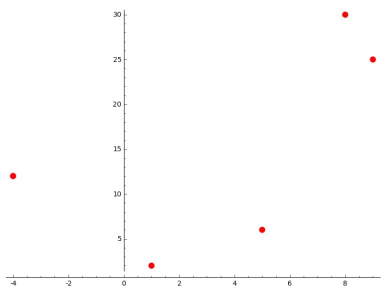
Polynomial interpolation is a method for solving the following problem:

> Given a set of n of data points with distinct x–coordinates $\{(x_i, y_i)\}_{i=1}^n$ find a polynomial of degree at most $n-1$ that passes through each point.

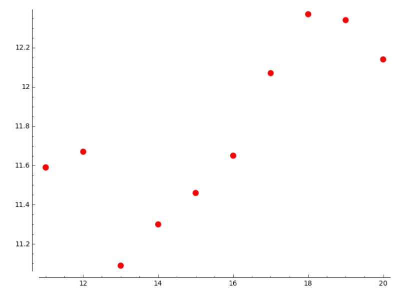
Example graphs of these polynomials for different data sets are shown below:



(A)



(B)



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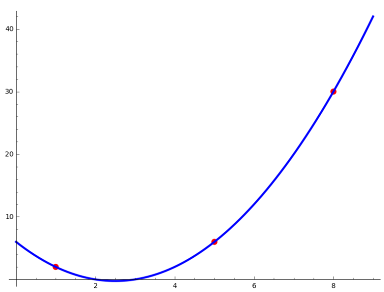

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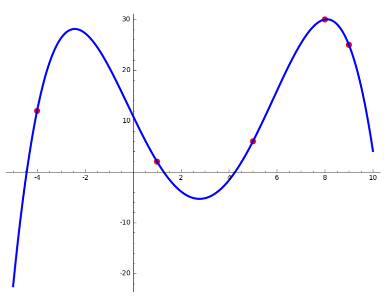

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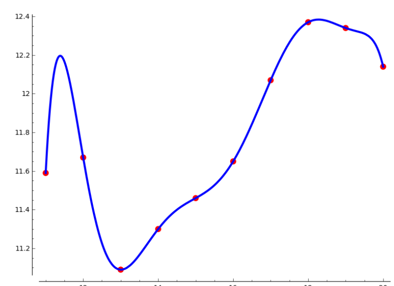

(F)

These interpolating polynomials provide a smooth approximation to the data points allowing for efficient extrapolation of the data. These methods are often used for constructing numerical approximations for computing complicated function values and evaluating differential equations. Methods for constructing interpolating polynomials go back hundreds of years. Any polynomial interpolation problem can be solved by realizing the coefficients as the solution to a linear system of the form:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

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The theory of Vandermonde determinants shows that this system has a unique solutions and hence there exists a unique interpolating polynomial: $c_{n-1}x^{n-1}+c_{n-2}x^{n-2}+\cdots+c_1x+c_0$. However, solving this system requires inverting a $n \times n$ matrix which can be computationally infeasible and can introduce distortions due to poor matrix conditioning. The method presented in class, which is described the section below, is due to a combination of Euler, Waring, and Lagrange, and was published in the late 1700's.

2. Formula

The Lagrange interpolation formula writes the interpolating polynomial for $\{(x_i, y_i)\}_{i=1}^{n}$ as a linear combination of n degree $n-1$ polynomials each of which is zero at exactly $n-1$ of the points. The polynomials are given by the general formula:

$$\ell_i = \prod_{\substack{0 \le j \le n: \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

and the interpolation polynomial is calculated as:

$$L(x) = \sum_{i=1}^{n} y_i \ell_i(x)$$

In Math 1 we looked specifically at the version where $n = 3$ so we obtain the following quadratic polynomials for $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$:

$$\begin{aligned}
\ell_1 &= \left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right) \\
\ell_2 &= \left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right) \\
\ell_3 &= \left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right) \\
L(x) &= y_1\ell_1 + y_2\ell_2 + y_3\ell_3 \\
&= y_1\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right) + y_2\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right) + y_3\left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right)
\end{aligned}$$

3. Example

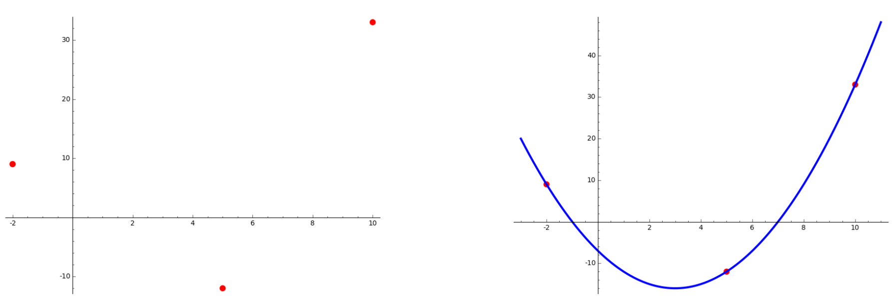
Fit a quadratic polynomial to the following data points $\{(-2,9),(5,-12),(10,33)\}$.

$$\begin{aligned}
\ell_1 = & \left(\frac{x-5}{-2-5}\right)\left(\frac{x-10}{-2-10}\right) \\
& \left(\frac{x-5}{-7}\right)\left(\frac{x-10}{-12}\right) \\
& \left(\frac{x^2-15x+50}{84}\right)
\end{aligned}$$

$$\begin{aligned}
\ell_2 = & \left(\frac{x+2}{5+2}\right)\left(\frac{x-10}{5-10}\right) \\
& \left(\frac{x+2}{7}\right)\left(\frac{x-10}{-5}\right) \\
& \left(\frac{x^2-8x-20}{-35}\right)
\end{aligned}$$

$$\begin{aligned}
\ell_3 = & \left(\frac{x+2}{10+2}\right)\left(\frac{x-5}{10-5}\right) \\
& \left(\frac{x+2}{12}\right)\left(\frac{x-5}{5}\right) \\
& \left(\frac{x^2-3x-10}{60}\right)
\end{aligned}$$

$$\begin{aligned}
L(x) = & \quad y_1\left(\frac{x^2-15x+50}{84}\right)+y_2\left(\frac{x^2-8x-20}{-35}\right)+y_3\left(\frac{x^2-3x-10}{60}\right) \\
= & \quad 9\left(\frac{x^2-15x+50}{84}\right)+-12\left(\frac{x^2-8x-20}{-35}\right)+33\left(\frac{x^2-3x-10}{60}\right) \\
= & \quad x^2-6x-7
\end{aligned}$$



(A) (B)

Department of Mathematics, Dartmouth College
E-mail address: `ddeford@math.dartmouth.edu`